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A topological approach to designing and constructing dynamical visual metaphors of multicultural and intercultural systems II-B

Una aproximación topológica al diseño y construcción de metáforas visuales dinámicas de sistemas multiculturales e interculturales II-B

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Resumen

En este ensayo continuamos nuestro trabajo previo en relación a intentar dotar a los estudios multiculturales e interculturales de un sólido fundamento topológico y de sistemas dinámicos. En primer lugar, nos aproximamos a las antiguas cosmovisiones taoístas y platónicas para mostrar que a pesar de sus diferencias, ambas comparte una estructura topológica común, a saber, $\{X, \Phi, A, B\}$, donde $A \cup B = X$ y $A \cap B = \Phi$. Más allá de argumentos metafísicos, este tipo de completitud disjunta es la estructura topológica básica que soporta al dualismo y a los abordajes dualistas de toda clase de fenómenos y problemas en ciencias y humanidades. El dualismo, además, conduce de forma natural a formular los modelos más sencillos de sistemas dinámicos multidimensionales no triviales. Luego de mostrar, a través de ejemplos y casos de estudio, que las topologías pueden ser pensadas como herramientas tanto de análisis como de diseño, procedemos a revisar algunos fundamentos de topología y sistemas dinámicos para darle soporte a la construcción de mapas de clasificación topológica para sistemas dinámicos de segundo orden, nuestra herramienta básica para modelar y clasificar los patrones de comportamiento cualitativos no equivalentes que un sistema dinámico dual, es decir, de segundo orden, puede exhibir. Hacia el final del ensayo regresamos al libro de leyendas de la Hermandad de los Monjes Azules y la Tribu de los Guerreros Escarlata, tomamos prestada de la biofísica la dinámica de las ecuaciones de Fitzhugh, y las utilizamos para mostrar todas las dinámicas genéricas locales que una tal sociedad bicultural Azul-Escarlata podría tener.

Palabras claves: Matemáticas, Topología, Sistemas Dinámicos, Modelos, Metáforas, Multiculturalismo, Interculturalismo.

Abstract

In this work we continue our previous work on essaying giving multicultural and intercultural studies some sound topological and dynamical systems foundation. We first approach ancient Taoist and Platonist cosmovisions to show that despite their differences both share a common topology, namely, $\{X, \Phi, A, B\}$ with $A \cup B = X$, and $A \cap B = \Phi$. Metaphysical arguments aside, this kind of disjoint completeness is the basic topological structure supporting dualism and dualist approaches in every sort of phenomena and problems in sciences and humanities. Dualism, moreover, naturally leads to formulate the simplest models for nontrivial multidimensional dynamical systems. After showing, through examples and case studies, that topologies may be thought of both as analysis or design tools, we proceed to review some basics of topology and dynamical systems to support the construction of topological classification maps for second-order dynamical systems, our basic tool for modeling and classifying all non-equivalent qualitative patterns of behavior a dual, that is to say, a second-order, dynamical system, may exhibit. Towards the end of the essay we go back to the book of legends of the ancient Brotherhood of the Blue Monks and the Tribe of the Red Knights, borrow Fitzhugh equations dynamics from biophysics, and uses it to show all the local generic dynamics such a Blue-Red bicultural society might have.

Keywords: Mathematics, Topology, Dynamical Systems, Models, Metaphors, Multiculturalism, Interculturalism.

1 Introduction

In the first part of this essay (Rodríguez-Millán 2020) we approached the problem of multicultural system through a sequence of examples and case studies chosen to show that despite deep cultural and historical differences Taoist and Platonist cosmovisions share a common topological structure, that wars can be designed to transform multicultural societies with richer topologies into bicultural societies with poorer topologies, or that failures in public services can transform efficient bicultural commerce systems into very inefficient multicultural commerce systems. In all this cases topology provides a first model of compartmentalization of societies into sets of disjoint subsets of equivalent citizens according to some kind of equivalence criterion. These intuitive ideas were afterwards formalized through the introduction of the concepts of equivalence relation, equivalence class, partition, topological space, quotient map, quotient topology, and quotient space. In plane words, however, this entire mathematical technicality just serves the purpose of identifying, in a precise way, the cast of the comedy we are interested either in deciphering or putting into scene. In the second part of this essay we will focus our attention into the interactions of representatives of the equivalence classes a multicultural system consists of. Metaphorically speaking we are now going to concentrate ourselves into the drama of multicultural systems, and will study in detail all possible real dynamics of the bicultural system of the brotherhood of the blue monks and the tribe of the red knights. We will leave the study of their complex dynamics for a next paper.

2 Topology, Linear Algebra, and Dynamical Systems

Reviewing the equivalence relation literature (Dugundji 1976, Kelley 1955, Munkres 1972) evidence that the application $p: X \to X/R$, which sends an element $x \in X$ into its equivalence class $[x]$, receives different names: the *projection*, the *quotient map*, the *identification map*, while the space X/R may be called the *quotient space of* X with respect to R (or modulo R), the *decomposition space of* X, or the *identification space of* X. According to (Munkres 1972) some mathematician call X/R the identification space of X *"for they think of* X/R *as having been obtained by identifying all the elements in each partition class to a single point"*. Identifying all the elements of an equivalence class to a single *representative* of the equivalence class means that, once the generating equivalence relation has been defined, topologists loose the ability to distinguishing between R-equivalent objects, or equivalently, all elements of an equivalence class look undistinguishable to them; in consequence, topologists just take any single representative of the equivalence class as the universal model of all R-equivalent objects. Consistently with this view, Kelley (Kelley 1955) defined topologists this way: "A topologist is a man who doesn't know the difference between a doughnut and a coffee cup".

From the practical point of view, it is a happy circumstance that topologists "think of X/R as having been obtained by identifying all the elements in each partition class to a single point", because that is exactly the same thing that the seller S of Case Study 2 (Rodríguez-Millán 2020) does in relation to the set C of his clients. Given that before the blackout all the clients are indistinguishable, the dynamics of the system (S, C) can be identified with the dynamics of the system $(S, \{c\})$, where the singleton $\{c\}$ is a single representative of the whole set of clients C. As already explained in Example 5 (Rodríguez-Millán 2020), after the blackout, the identification space of C explode into $\{C, M, E, F, G\}$, and the dynamics of the system into the dynamical system $(S, \{e\}, \{f\}, \{g\})$, where the singletons $\{e\}, \{f\}, \{g\}$ are the representatives of the identification classes E, F, G, of clients, respectively.

The very foregoing argument can be applied to all previous examples and cases study … even tough we might be accused of topologists, that is to say, of not knowing the difference between doughnuts and coffee cups.

2.1 Linear Dynamical Systems

Even though we could define dynamical systems intrinsically as in (Hirsch-Smale 1974, Arnol'd 1982, Bröcker-Jänich 1982), for the sake of the present work, we will follow the traditional approach, and will use linear differential equations to model linear dynamical systems.

Definition 1. An *n-dimensional linear dynamical system* is any process admitting being modeled by an nth-order linear differential equation

$$\dot{x} = Ax + Bu(t), x(0) = x_0, \tag{1}$$

where A and B are real matrices of appropriate dimensions, $x \in \mathbb{R}^n$, and $u(t) \in \mathbb{R}^m$, for every $t \in \mathbb{R}$.

As it is well known (Hirsch-Smale 1974), the *trajectories* of the linear dynamical system (1) are:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\sigma)}Bu(t)dt = x_f(t) + x_u(t), \tag{2}$$

where $x_f(t) = e^{At}x_0$, the solution of the homogeneous system

$$\dot{x} = Ax, x(0) = x_0, \tag{3}$$

is called the *free dynamics* of system (1), and $x_u(t) = \int_0^t e^{A(t-\sigma)}Bu(t)dt$, called the *forced dynamics* of system (1), models the action of the *external force* $u(t)$ on the dynamics of the system. The free dynamics $x_f(t) = e^{At}x_0$, can be additively decomposed as a linear combination of the *fun-*

damental solutions $x_{fi}(t) = e^{\lambda_i t}$ of (3), where the parameters $\lambda_i, 1 \le i \le n$, are the roots of the *characteristic polynomial*

$$p_A(\lambda) = Det(A - \lambda I), \tag{4}$$

of matrix A. The roots $\lambda_i, 1 \le i \le n$, are called the *eigenvalues*, or the *characteristic values*, of matrix A.

The origin $0 \in \mathbb{R}^2$ is a constant solution of $\dot{x} = Ax, x(0) = x_0$, called *equilibrium point*, which organizes the long-run behavior of the set of trajectories. The origin is a *sink* (*source*) if all the eigenvalues of A have negative (positive) real part, in which case the trajectories converge to (diverges from) it. Sinks (sources) are also called *stable (unstable) equilibria*. When one eigenvalue is positive and the other is negative, the origin is called a *saddle* point.

Eigenvalues are algebraic invariants (Hirsch-Smale 1974) under linear transformations of coordinates, i.e., the set of eigenvalues of matrices A and QAQ^{-1} are the same, for every invertible matrix Q. So, without loosing generality, we will assume that systems are given in Jordan canonical forms, with respect to the basis of the eigenvectors associated to the eigenvalues of matrix A.

Case Study 1. Linear Second-Order Dynamical Systems. Let us consider the second-order homogeneous linear system $\dot{x} = Ax, x(0) = x_0$, with $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. The characteristic polynomial of the matrix A is

$$p(\lambda) = Det(A - \lambda I) = \lambda^2 - Tr(A)\lambda + Det(A), \tag{5}$$

where $Tr(A) = a_{11} + a_{22}$, and $Det(A) = a_{11}a_{22} - a_{12}a_{21}$. Then, the eigenvalues of matrix A are

$$\lambda_{1,2} = \frac{1}{2}\left(Tr(\mathrm{A}) \pm \sqrt{(Tr(\mathrm{A}))^2 - 4Det(\mathrm{A})}\right), \tag{6}$$

which are the very same roots of the second-degree polynomial $p(\lambda)$ in Example 2 (Rodríguez-Millán 2020), under the identification of parameters, $a = -Tr(A)$, and $b = Det(A)$. So, the partition of the set of second-degree polynomials in Figure 3 also holds for the second-degree characteristic polynomials of second-order linear systems (3), if we rotate Figure 3 (Rodríguez-Millán 2020) around the vertical a-axis or, equivalently, if we interchange the red and green compartments in Figure 3 (Rodríguez-Millán 2020). ■

To know the eigenvalues of the linear homogeneous system (3) allows constructing the topological classification map of all possible free dynamics of the system, up to linear transformations of coordinates. The equivalent relation supporting the construction of topological classification maps is resumed in Table 1, where we have respected the color code of the quotient space in Figure 3 (Rodríguez-Millán 2020). The identification map associated to this equivalent relation permits equipping topological classification maps with the quotient topology, transforming them into quotient spaces.

Table 1. Equivalence relation supporting the construction of the topological classification map of a second-order linear dynamical systems.

$Tr(A)$	$Det(A)$	$\Delta(A)$	Topology	Code
	< 0		Saddle	(U, S)
Sources				
> 0	> 0	> 0	Nodes	(U, U)
> 0	> 0	0	Focus	(U, U)
> 0	> 0	0	Improper Nodes	(U, U)
> 0	0	> 0	Degenerated Equilibria	(C, .)
> 0	> 0	< 0	Spirals	(@, U)
Sinks				
< 0	> 0	> 0	Nodes	(S, S)
< 0	> 0	0	Improper Nodes	(S, S)
< 0	0	> 0	Degenerated Equilibria	(C, .)
< 0	> 0	< 0	Spirals	(@, S)
0	> 0	< 0	Centers	(@, C)

In Table 2 we collect the Jordan canonical form associated to all different eigenvalue configurations of matrix A, and all possible forms of the state-transition matrix e^{At}.

Table 2. Jordan canonical forms of matrix A, and the associated fundamental solutions e^{At}. Even thought both A and e^{At} had the same formal structure, the red-green color code in the column of the eigenvalues indicates whether a given topology represents a sink or a source, consistently with the color code in Table 1 above.

Topology	A	e^{At}	Eigenvalues
Saddle	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	$\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$	$\lambda_1 \in \mathbb{R}_-$ $\lambda_2 \in \mathbb{R}_+$
Nodes	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	$\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$	$\lambda_1, \lambda_2 \in \mathbb{R}_-$ $\lambda_1, \lambda_2 \in \mathbb{R}_+$
Focus	$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix}$	$\lambda \in \mathbb{R}_-$ $\lambda \in \mathbb{R}_+$
Improper Nodes	$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$	$\lambda \in \mathbb{R}_-$ $\lambda \in \mathbb{R}_+$
Degenerated Equilibrium	$\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} e^{\lambda t} & 0 \\ 0 & \kappa \end{pmatrix}$	$\lambda \in \mathbb{R}_-$ $\lambda \in \mathbb{R}_+$
Spirals	$\begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix}$	$e^{\sigma t}\begin{pmatrix} cos\omega t & -sin\omega t \\ sin\omega t & cos\omega t \end{pmatrix}$	$\sigma \in \mathbb{R}_-$ $\sigma \in \mathbb{R}_+$
Centers	$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$	$\begin{pmatrix} cos\omega t & -sin\omega t \\ sin\omega t & cos\omega t \end{pmatrix}$	$\sigma = 0$ $\omega \in \mathbb{R}_+$

2.2 Nonlinear Dynamical Systems

Definition 2. A *smooth n-dimensional nonlinear dynamical system* (NLDS) is any process admitting being modeled by an nth-order nonlinear differential equation

$$\dot{x} = f(x), x(0) = x_0, \tag{7}$$

where $x \in \mathbb{R}^n$, and $f \in C^1[\mathbb{R}^n, \mathbb{R}^n]$.

In definition above we restricted the class of vector-fields f to those of class C^1, to assure the initial-value problem (7) has a unique solution. Less restrictive definitions of NLDS may be given (Coddington-Levinson 1956).

Definition 3. A point $X \in [\mathbb{R}, \mathbb{R}^m]$ is called an *equilibrium point* of the NLDS (7) if $f(X) = 0$. The linear dynamical system

$$\dot{x} = D_x f(X)\,(x - X),\ x(0) = x_0, \tag{8}$$

is called the linearization of the NLDS (7) around the equilibrium point X. If all the eigenvalues of $D_x f(X)$ have non-zero real parts, X is called a *hyperbolic equilibrium* point.

From the mathematical point of view, the most important result of the theory of NLDS may surely be the Theorem of Existence and Uniqueness of Solutions (Coddington-Levinson 1956, Hirsch-Smale 1974) but, from the point of view of the applications of the theory of NLDS, perhaps the most important result is *Hartman-Grobman theorem* (Perko 1996, Hartman 1982) which assures that as far as the Jacobian matrix $D_x f(X)$ has no eigenvalues with zero real parts, the local dynamics of the NLDS (7) and its linearization (8) are homeomorphically equivalent in a small enough neighborhood of the equilibrium point X. So, Hartman-Grobman theorem fully supports constructing the topological classification map of the linearized system (8) around the hyperbolic equilibrium point X, and then use such results to describe the local dynamics of the NLDS (7) around X, because both dynamics are homeomorphically identical in an small enough neighborhood of X.

The exhaustive description of the global dynamics of a NLDS may be an extremely difficult and complex task, which in most cases easily leads to unanswered mathematical questions. Yet, even the longest trip start with a first step, and that is precisely what the topological classification of equilibrium points represents in the analysis of the dynamics of a NLDS; we do not even pretend to cross this border in the present essay.

Case Study 2. A Cultural Approach to Block Diagrams. Systems theory uses several languages to approach system analysis from different perspectives: the analytic language of differential equations to model deterministic systems; the language of geometry and topology to classify the qualitative behavior of sets of trajectories; the language of algebra providing canonical representations structurally fitted to trap properties like stability, controllability, observability; the spectral language allowing to think of control systems like filters operating on signals; and yet, the language of block diagrams showing up the internal wiring of the states of the systems, and therefore their patterns of mutual interdependence and influence.

From the examination of Jordan canonical form in Table 2 it follows that there exist only tree non-equivalent wiring schemes: the *perpendicular topology*, associated to saddles, nodes, foci and degenerated equilibrium; the *circular topology*, associated to spirals and centers; and the *tangential topology* associated to the improper nodes, which represents a kind of transition topology between the perpendicular and the circular topologies.

If we think of the second-order dynamical system (9)

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = A\begin{pmatrix}x_1\\ x_2\end{pmatrix} \tag{9}$$

as an abstract model of the dynamics of representatives x_1 and x_2 of two equivalence classes of citizens a society consists of, the block diagrams of Figure 1 clearly establish that there are only three canonical types of interrelations between citizens x_1 and x_2: (i) the behaviors of x_1 and x_2 are completely independent from each other:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}, \tag{10}$$

as in Figure 1-a; (ii) citizen x_2 dominates x_1 unidirectionally:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}, \tag{11}$$

as in Figure 1-b; and (iii) citizens x_2 and x_1 influence each other equipotently:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}\sigma & -\omega\\ \omega & \sigma\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}, \tag{12}$$

as in Figure 1-c, generating emerging oscillatory patterns of behavior with no parallel amongst first-order dynamical systems. So, we could think of second-order dynamical systems (10) to (12) as canonical models of isolationism, unidirectional interventionism, and reciprocal influences, respectively. Isolationism means that, as block diagram of Figure 1-a suggests and the fundamental matrix e^{At} in Table 2 confirms, the trajectories (behaviors) of citizens x_2 and x_1 are completely independent of each other. Unidirectional interventionism means that the behavior of citizen x_2 is independent of x_1, yet it intervenes and modifies the dynamics of citizen x_1. The counterclockwise closed-loop containing the two green blocks in the block-diagram of Figure 1-c is the physical implementation of the rotation matrix e^{At} associated to the centers in Table 2, while the yellow blocks are the physical counterparts of the exponential term $e^{\sigma t}$ in the fundamental matrix e^{At} associated to the spirals in Table 2.

We close this case study with a final algebraic comment. If eigenvalues λ_1, λ_2 are the roots of the characteristic polynomial of matrix A, then

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 + (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2. \quad (13)$$

If we compare this model of the characteristic polynomial with its previous expression in (5), it comes out that:

$$Det(A) = \lambda_1\lambda_2 \quad (14)$$

$$Tr(A) = \lambda_1 + \lambda_2 \quad (15)$$

$$\Delta(A) = (\lambda_1 - \lambda_2)^2, \quad (16)$$

three algebraic invariants for linear second-order dynamical systems. When the eigenvalues λ_1, λ_2 are equal, $\Delta(A) = 0$, and this is precisely the frontier separating the isolationist systems of Figure 1-a, from the systems of reciprocal influences of Figure 1-c. $\Delta(A) = 0$ characterizes the unidirectional interventionist systems of Figure 1-c. ■

3 Case Study 3. Blues Monks vs. Reds Knights

Up until now we have developed a few elements of the languages of topology, linear dynamical systems, and block diagrams needed to model and describe the dynamics of multicultural systems. So, we count now on first-order linear dynamical systems to model the behavior of individuals, and second-order linear dynamical systems to model the interactions between pairs of individuals. Table 2 collects and classifies all possible patterns of interaction between individuals. Moreover, Hartman-Grobman theorem allows using linear dynamical systems as local models of nonlinear dynamical systems, around hyperbolic equilibrium points. In this section we will go back to the model of the first bicultural society ever, consisting of two equivalence classes: the Brotherhood of the Blue Monks, and the Tribe of the Red Knights. For academic purposes we will suppose the interactions between the Blue Monks and the Red Knights (Rodríguez-Millán et al 2019) are governed by Fitzhugh equations, a well-known biophysical model for the generation of electrical signals in electrically excitable cells. Fitzhugh's model is kind of an innocent, helpless looking system, yet it possesses a rich, and complex dynamics, with several types of bifurcations and nontrivial periodic orbits.

We will assume (Rodríguez-Millán 1992) that Fitzhugh equations are given as:

$$\dot{x} = I + x + y - \frac{1}{3}x^3 \quad (17\text{-a})$$
$$\dot{y} = \mu(a - x - by). \quad (17\text{-b})$$

Let $z = I + y$ and $\varphi = a + bI$. In this coordinates the Fitzhugh equations become

$$\dot{x} = y + x - \frac{1}{3}x^3 \quad (18\text{-a})$$
$$\dot{y} = \mu(\varphi - x - by). \quad (18\text{-b})$$

If $b \in [0,1]$, Fitzhugh equations have a unique equilibrium point $x_0(\varphi)$ for every $I \in (-\infty, \infty)$. This unique equilibrium point is the solution of the third-degree polynomial $\frac{1}{3}x^3 + \left(\frac{1}{b} - 1\right)x - \frac{1}{b}\varphi = 0$. The linearization of equations (18) around the unique equilibrium point $x_0(\varphi)$ is:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}1 - x_0^2(\varphi) & 1\\ -\mu & -b\mu\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = K\begin{pmatrix}x\\ y\end{pmatrix} \Rightarrow, \quad (19)$$

$$Tr(K) = 0 \Leftrightarrow \mu = \frac{1}{b}(1 - x_0^2). \quad (20)$$

$$Det(K) = 0 \Leftrightarrow \mu = 0 \ \ or \ \ x_0^2 = 1 - \frac{1}{b} \quad (21)$$

$$\Delta(K) = 0 \Leftrightarrow \mu = \frac{2 - b(1 - x_0^2) \pm 2\sqrt{1 - b(1 - x_0^2)}}{b^2}. \quad (22)$$

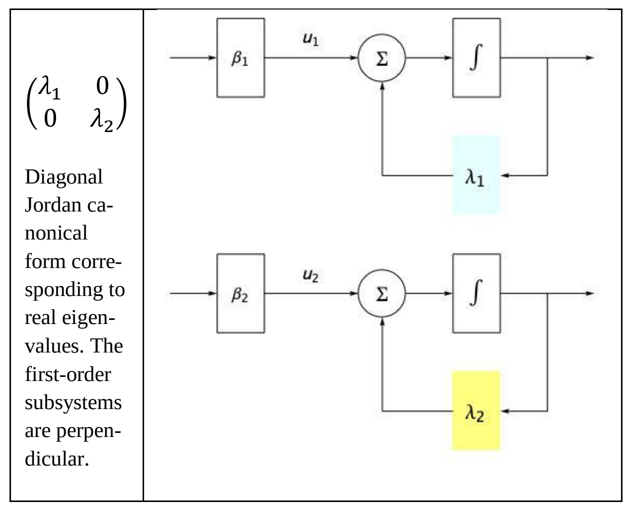

Figure 1-a. Block diagram of all diagonalizable second-order linear dynamical systems with real eigenvalues, including the diagonalizable non-generic case of multiplicity 2 eigenvalues $\lambda_1 = \lambda_2$.

Let us define the following two functions:

$$\mu_-(b, x_0) = \frac{1}{b^2}\left(2 - b(1 - x_0^2) - 2\sqrt{1 - b(1 - x_0^2)}\right) \quad (23)$$

$$\mu_+(b, x_0) = \frac{1}{b^2}\left(2 - b(1 - x_0^2) + 2\sqrt{1 - b(1 - x_0^2)}\right). \quad (24)$$

The zero-level curves (20), (21), and (23-24), of the trace, the determinant, and the discriminant, respectively, are the conceptual elements we need to construct the topological classification map of the linear dynamical system (19). This map will provided an exhaustive description of the local dynamics of the Fitzhugh equations around its unique equilibrium point located at the origin, when $b \in [0,1]$.

$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

Jordan canonical form corresponding to a double real eigenvalue with nontrivial nilpotent component, which destroys the perpendicularity of the two first order subsystems.

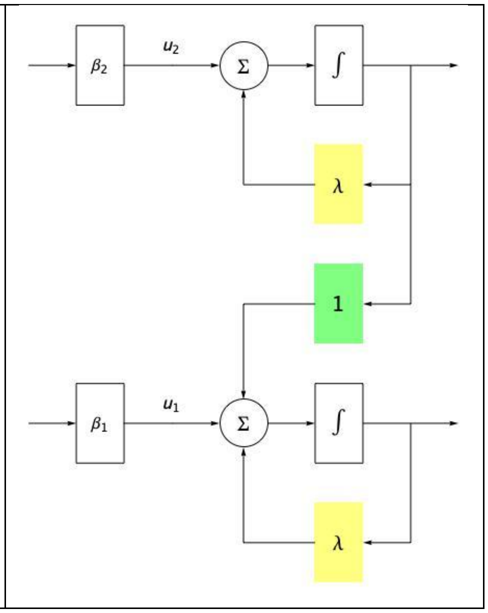

Figure 1-b. Block diagram representation of a second-order dynamical system with a double real eigenvalue, and nilpotent component. The entry "1" that destroys diagonality, also destroy the orthogonal intersection of the two one-dimensional subsystems. Nilpotency does not depends at all on the value on the entry "1" in the Jordan canonical form. Any other value would also perform well.

$\begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix}$

Jordan canonical form associated to a pair of complex conjugate eigenvalues, generating a loop in the block diagram.

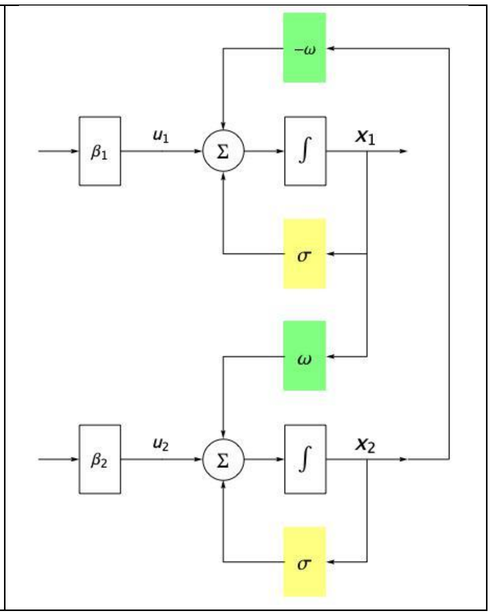

Figure 1-c. Block diagram of a second-order dynamical system with a pair of complex conjugate eigenvalues. If we compare this block diagram with the two previous ones, we could think of the three of them as a sequence of systems evolving out of a state of asymmetric disconnection, into another state of full symmetric connection. The two first-order subsystems in Figure 1-a are disconnected with $\lambda_1 \neq \lambda_2$, while in the two first-order subsystems in Figure 1-b, $\lambda_1 = \lambda_2$, but the upper subsystem unidirectionally influence the behavior of the lower subsystem. A second backwards interconnection appears in the complex conjugate case shown in this figure, destroying the unidirectionality of influences, and permitting the mutual equilibrated interdependence of the behaviors of the two first-order subsystems.

Figure 2-a shows the signs classification map for $Tr(K)$ in (20). As $Tr(K) > 0$ in the yellow zone, the associated fundamental solutions grow as $t \to \infty$.

Figure 2-b shows the zero-level curve of the determinant function $Det(K)$, and its map of signs. $Det(K)$ is negative in the cyan zone, and therefore for all pairs (x_0, μ) in this zone the equilibrium point is a saddle point. Given that $Det(K)$ is positive in the terracotta zone, the topology of the equilibrium point is manifold depending on the values of $Tr(K)$ and $\Delta(K)$.

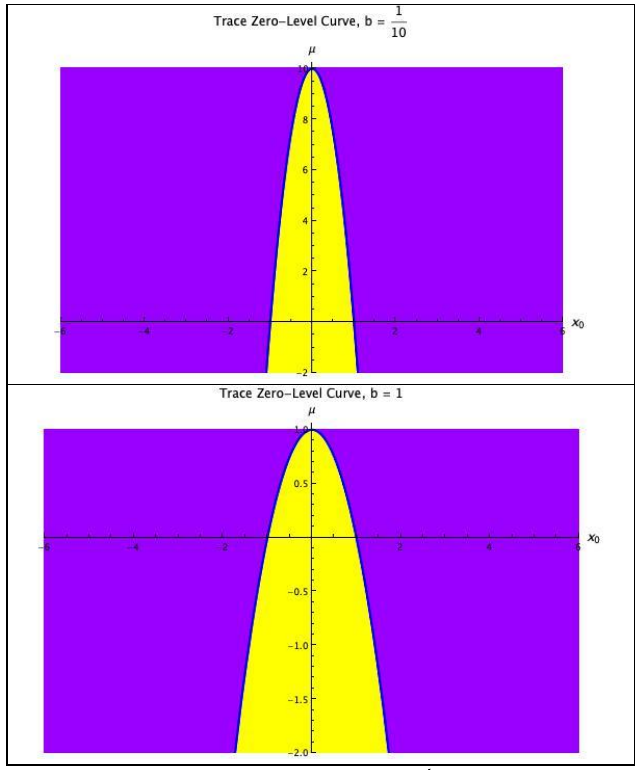

Figure 2-a. Zero-level curve of $Tr(K)$, for $b = \frac{1}{10}, 1$, in the natural order. The yellow (purple) zones correspond to positive (negative) values of the trace. So, for ordered pairs (x_0, μ) of values of the parameters in the yellow (purple) zone, the fundamental solutions diverge from (converge to) the equilibrium point. Then, for all values of the parameters x_0 and μ in the yellow (purple) zone the origin is an *unstable (stable) equilibrium point*.

In Figure 2-c we show the zero-level curves of the discriminant and the associated map of signs. In this case the discriminant function has two zero-level curves: the lower curve $\mu_-(b, x_0)$ and the upper curve $\mu_+(b, x_0)$, defined as in (23) and (24), respectively, making the sign map a bit more complicated. The upper zero-level curve $\mu_+(b, x_0)$ is very sensitive to perturbations of the parameter $b \in [0,1]$: for $b \longrightarrow 0$, the curve $\mu_+(b, x_0)$ moves upwards and its minimum $\mu_+(b,\ 0) \longrightarrow \infty$. Reversely, as $b \longrightarrow 1$, $\mu_+(b, x_0)$ moves downwards while $\mu_-(b, x_0)$ moves upwards, intersecting each other at the point $\mu_+(1, 0) = \mu_-(1, 0) = 1$, when $b = 1$

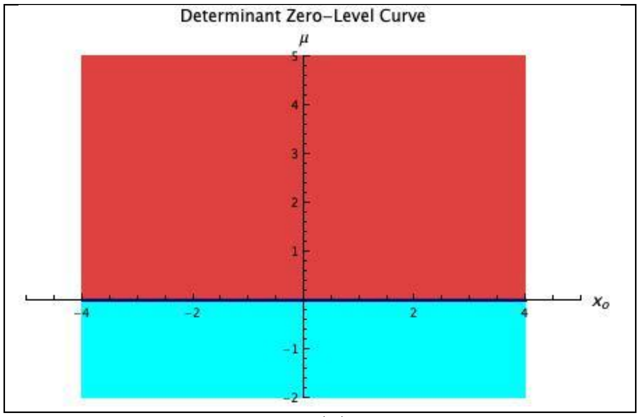

Figure 2-b. Zero-level curve of $Det(K)$, which is invariant for every $b \in [0,1]$. The cyan zone corresponds to negative values of $Det(K)$, and therefore the system has a saddle equilibrium point at the origin. For pairs of values of the parameters (x_0, μ) in the terracotta zone $Det(K) > 0$, and the equilibrium point may have different topological structures, depending on the values of the trace, the determinant, and the discriminant.

With all the required elements at hand, let us proceed to construct the topological classification map of the equilibrium point of the second-order system of the Blue Monks and the Red Knights. We will think of the topological classification map as a b-parameterized image, because Figures 2-a and 2-c clearly show that the maps of signs of $Tr(K)$ and $\Delta(K)$ are b-parameterized, then the topological classification map also. We construct first the topological classification map for $b = \frac{1}{2}$, and proceed then to run a stop-motion video for $b \in [0,1]$.

Step 1: Saddles. When $\mu < 0$, $Det(K) < 0$, and the equilibrium point is a saddle. We may forget about the upper half of Figure 2-b.

Step 2: Spirals. According to Table 1, for every (x_0, μ) belonging to the yellow (purple) zone of the map of signs of $Tr(K)$, the equilibrium point is a source (sink), while according to Figure 2-c the green zone corresponds to spirals and the pink zone to nodes. So, yellow spirals are repelling spirals, while purple spirals are attracting spirals. We will preserve the yellow (green) color for the repelling (attracting) spirals.

Step 3: Nodes. Pink zone in Figure 2 represent nodes, while purple zone in Figure 2 represents sinks. So, purple nodes are attracting nodes, while yellow nodes are repelling nodes. We preserve the pink color for the attracting nodes, and introduce the orange color for the repelling nodes.

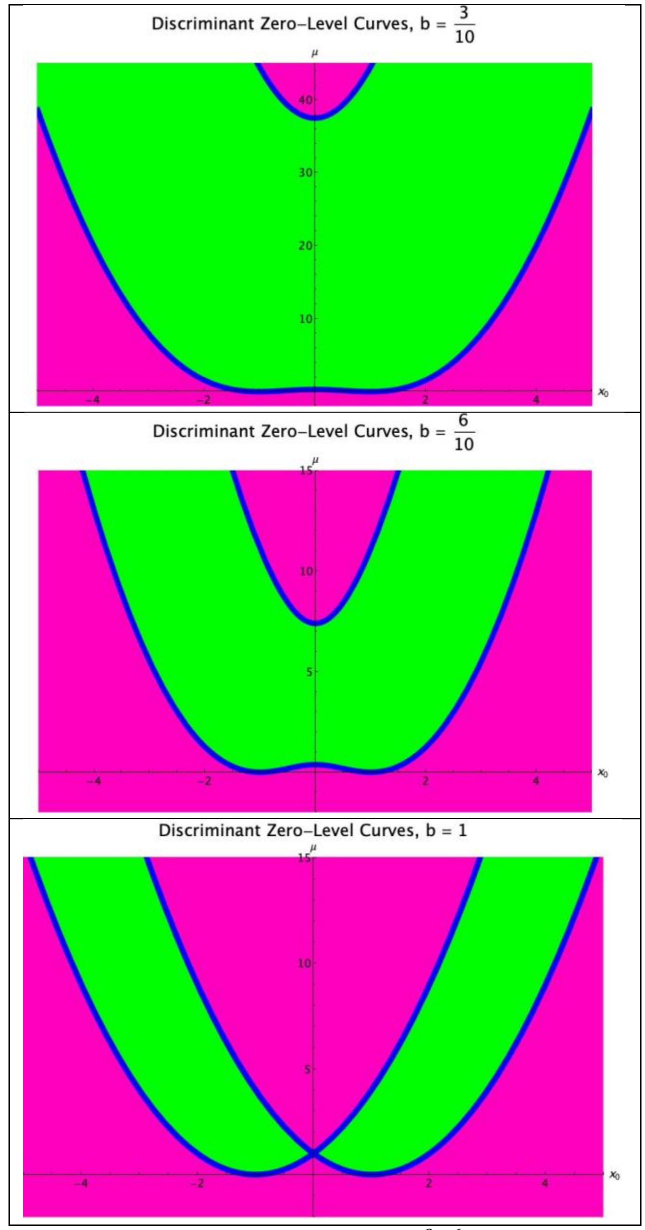

Figure 2-c. Zero-level curves of $\Delta(K)$, for $b = \frac{3}{10}, \frac{6}{10}, 1$. The discriminant $\Delta(K)$ is negative in the green zone, wherefrom the equilibrium point is either a center or a spiral for each pair (x_0, μ) of parameters in this zone. In the pink zone $\Delta(K) > 0$, then for every (x_0, μ) in the pink zone, the equilibrium point is a node. The blue zero-level curves are the boundaries between the subsets of systems with real eigenvalues and complex conjugate eigenvalues, being then associated to the improper nodes of Table 2, and the block-diagrams of Figure 2-b.

The topological classification map of the equilibrium point of the second-order dynamical system modeling the society of the Blue Monks and the Red Knights, with the borrowed academic dynamics of Fitzhugh equations (18-19), is shown in Figure 3. The topological classification map changes for different values of $b \in [0,1]$. These changes are collected in the stop-motion of Figure 4.

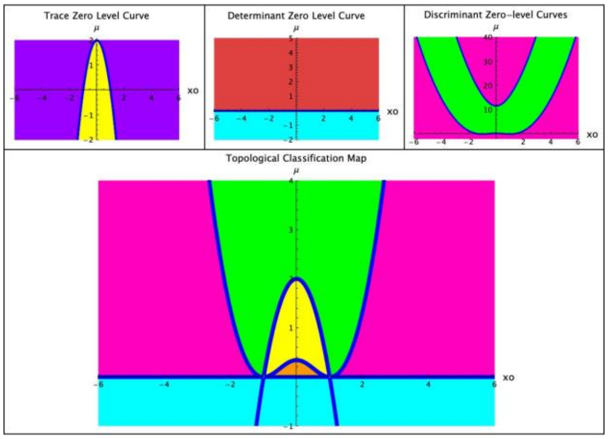

Figure 3. Topological classification map of the equilibrium point of Fitzhugh equations, for $b = \frac{1}{2}$. According to the color code stablished in the three-steps algorithm above the topological classification map should be read as follows: yellow = repelling spirals, green = attracting spirals, orange = repelling nodes, pink = attracting nodes, cyan = saddles.

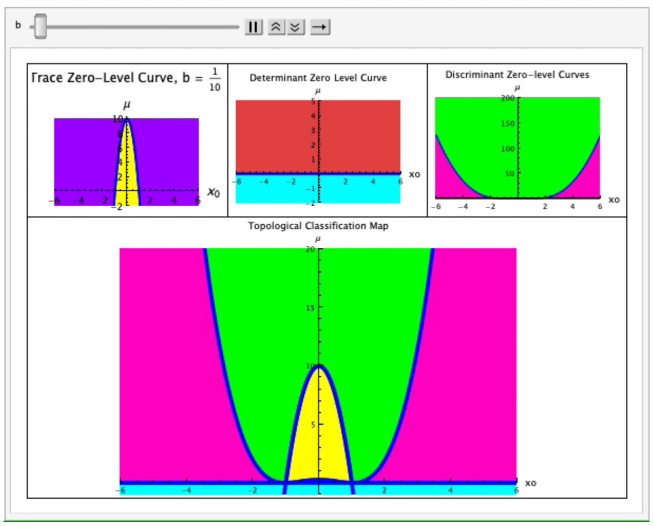

Figure 4. *b*-parameterized topological classification map of Fitzhugh equation, for $b \in \{\frac{1}{10}, \frac{2}{10}, \ldots, 1\}$.

In the Art Gallery of Figure 5, we show some of the local generic dynamics around the unique equilibrium point of Fitzhugh equation, when $b \in [0,1]$. Each image in Figure 5 is a representative of the equivalence class of the dynamics associated to one compartment of the topological classification map of Figure 3, as indicated in the legends of each image. The plotting times were chosen long enough to exhibit the asymptotic behaviors, yet short enough to generate visually appealing images.

Strictly speaking, the sequence of images below describes the behavior of the trajectories of the linearization (19) of Fitzhugh equation, around the origin. Yet, according to Hartman-Grobman theorem, these images are locally qualitatively (homeomorphically) identical to the set of trajectories of the nonlinear Fitzhugh equations (18) around their unique equilibrium point $x_0(\varphi)$.

Given that it may be inadequate and polemic to speak about "interactions" between the Blue Monks and the Red Knights in connection to the topological classification map of Figure 3, it would be perhaps more convenient to think of the topological classification map as a kind of map of "environmental conditions" that determine the simultaneous evolution of both groups. If we approach the problem this way, the topological classification map would tell us under what conditions both Blue Monks and Red Knights will grow, under what conditions both groups will decay, what are the conditions that would propitiate the growth of one group and the decay of the other, and, moreover, will also provided us information about the temporal patterns of growth or decay. In general, but very particularly in systems with several equilibrium points, grow and decay should in general be understood as to diverge from or converge to an equilibrium point. In the case of the Blue Monks and the Red Knights, for instance, the environmental conditions of the yellow and orange compartments propitiate the simultaneous "growth" to both Reds and Blues, conditions in compartments green and pink favor the "decay" of both groups, and the "cultural climate" of the cyan compartment favour the blooming of Blue culture and the stagnation of the Red culture, or vice versa. Yet, orange and yellow growth are different, for orange is kind of a linear growth, while yellow growth is kind of up and down or "oscillatory". Likewise for the pink and the green compartments, respectively. Even though the case study of the Blue Monks and the Red Knights is a pure academic example, supported by the Fitzhugh equations, it clearly suggests that it would be possible to model the qualitative dynamical behavior of non-physical soft systems. Going deeper into modeling particular human systems would of course require a detail modeling work, which is out of the scope of the present work.

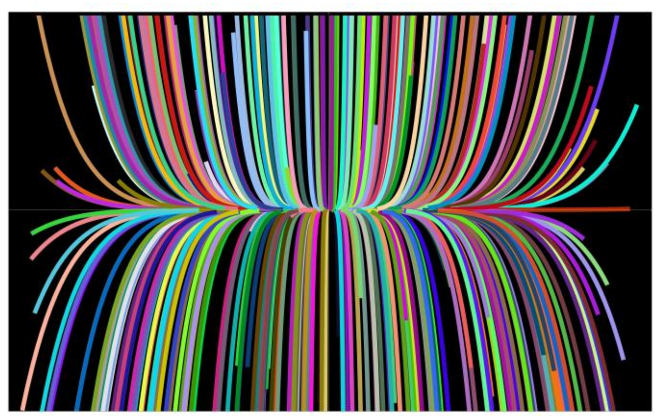

Figure 5-a. Attracting Node associated to the $(b, x_0, \mu) = \left(\frac{1}{2}, -4, 1\right)$. According to the topological classification map of Figure 3, this point is located in the left pink compartment, wherefrom it is an attracting node.

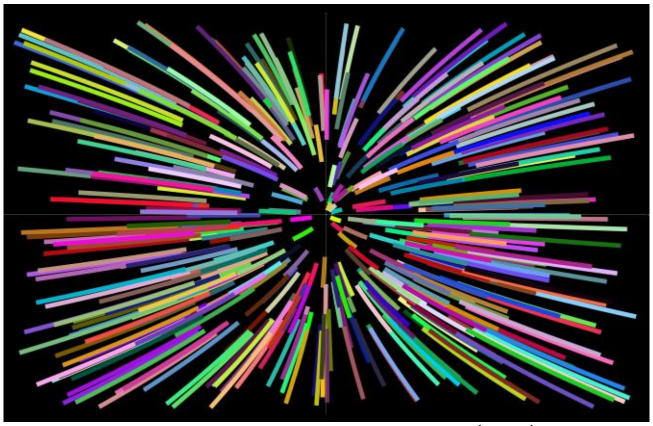

Figure 5-b. Repelling Node associated to $(b, x_0, \mu) = \left(\frac{1}{2}, 0, \frac{1}{3}\right)$. In the topological classification map, this point is localized in the small orange central open compartment, the equivalence class of the repelling nodes.

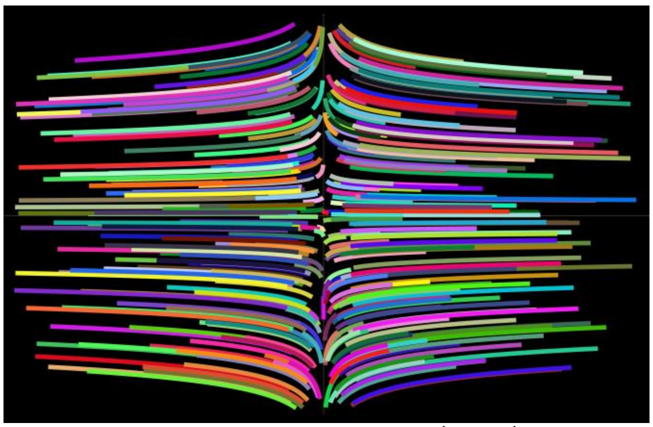

Figure 5-c. Saddle point associated to $(b, x_0, \mu) = \left(\frac{1}{2}, 0, -\frac{1}{3}\right)$. This point belongs to the lower cyan compartment, the open set of the quotient topology associated to the saddle points.

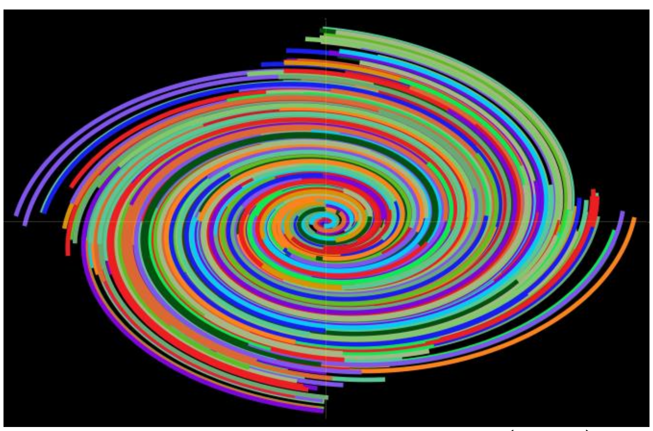

Figure 5-d. Repelling spiral associated to $(b, x_0, \mu) = \left(\frac{1}{2}, -\frac{1}{2}, 1\right)$. This point belongs to the yellow compartment of the topological classification map of Figure 3, corresponding to counterclockwise repelling spirals.

Figure 5-e. Attracting spiral associated to $(b, x_0, \mu) = \left(\frac{1}{2}, \frac{3}{2}, 1\right)$. This point is located in the green compartment of the topological classification map, associated to clockwise attracting spirals.

4 Conclusions

4.1 Dualism, Multiculturalism, and Topologies

Metaphysical arguments aside, dualism has historically proved to be a very efficient tool to describe and model all those systems whose dynamics essentially depends on two actors. The success of dualism is topological associated to equipping the supporting space X of the studied phenomena with a topology $\mathfrak{J} = \{A, C, X, \Phi\}$ consisting of two open sets A and C, such that $A \cup C = X$, and $A \cap C = \Phi$. The disjoint partition $\{A, C\}$ is the simple most topological structure capable to support the "dialog" between representatives of A and C, as equivalence classes, where a "dialog" is nothing but the most primitive formulation of what in contemporary mathematical language is called a second order dynamical system. Dialog pursues agreements, that is to say, permanent settle downs, i.e., equilibrium points, among actors in conflict. In modern notation, the first model for the dynamics of a second-order system is a linear second-order dynamical system, whose study sooner or later converges to the exhaustive analysis of its characteristic polynomial, which eventually leads to construct, either implicitly or explicitly, the topological classification map of the system, describing all possible different qualitative dynamics.

It is crystal clear that not every phenomenon admits being modeled as a second-order dynamical system, what leads to introduce bigger topologies with more open sets, higher order associated dynamical systems to model the dynamics of the systems, and higher degree polynomials to generate all the fundamental solutions of the systems. The complexity of this problem increases enormously with the number of equivalence classes of the systems and the order of the associated dynamical system. This is the world of multiculturality, where single cultures coexist but do not intermingle. Topological partitions lead to quotient topologies and quotient spaces, but impede interculturality.

For interculturality to emerge, the topology of X must contain open sets with non-trivial intersections. This would also obviously lead to richer "dialogs" and "forms of dialog", or equivalently, to more complex dynamical systems, necessarily defined on topological spaces with richer topological structures. So, at least from the strictly mathematical point of view, it is natural to expect that multicultural structures would be preferred to intercultural structures, even if it were just for complexity reasons, not to mention the required levels of intellectual sophistication required to approach interculturalism interculturally. This is a very interesting topic deserving deeper effort and study.

4.2 A Teaching Corollary

From the dualist, yet complementary, understanding vs. calculating or qualitative vs. quantitative paradigms, one may derive an important practical academic consequence, which is the permanent need to conceptually separate, and yet keep conscious of the interconnections between, the topological, the dynamical, the analytical, the algebraic, and the wiring approaches involved in the study of systems dynamics, because each one of these approaches traps different aspects of the structure and the dynamics of complex systems.

4.3 Order Increase, Complexity and Symmetries

Since the Big-Bang, the Universe has permanently been producing more and more complex structures (Harari 2017), which in many cases can be modeled through dynamical systems of increasing order and complexity. New emerging dynamics use to be associated to evolving towards more organized and symmetric structures. The sequence of block-diagrams of Figure 1 is an abstract elementary example of this evolutionary process. The study of Jordan canonical forms (Hirsch and Smale 1974) and the step-by-step construction of the associated block-diagrams is an interesting exercise in this respect, with an additional clear aesthetic component. The harmonic complementarity of behavioral, geometric, topologic, algebraic, and block diagram languages of system theory is, in my view, a beautiful example of harmony in nature. Well-done work is rewarded through the perfect matching of the pieces in the puzzle of the Universe.

4.4 Topological Explosions, Failures, and Perestroikas

In engineering applications, failures and structural changes are sometimes approached through their effects or manifestations in the dynamics or the frequency response of the systems. Yet, it would seem to be that the structural changes, failures, and perestroikas could be traced back to changes in the topological structure of the systems, which in turn modify the order and the structure of the associated dynamical systems representation of the studied phenomenon. This alternative view of failures deserves deeper attention and thought.

4.5 Art, Humanities, Sciences and Engineering

Art and humanities have a lot to offer to the modeling of complex systems because of their complementary mental and intellectual paradigms, and the additional degrees of freedom they have with respect to the rigidity of physics and mathematics. However, arts and humanities on the one hand, and sciences and engineering on the other hand, are a good representative of unfortunate academic topological partition. The boundaries preserving this status quo should be bored to promote cross-fertilization and mutual support between arts, sciences, humanities, and engineering. Much multicultural and intercultural work should be done in this direction at the universities and art academies around the world.

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