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Decidability between recognizable K-subsets

Decidibilidad entre K-subconjuntos reconocibles

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Abstract

This research deals with the problem of extending the conventional finite automata theory to the study of the equality theorem. For this purpose, an algebraic approach centered on the concepts of semi-rings, recognizable K-subsets and K-Σ-automata is proposed. The decidability of any pair of recognizable K-subsets is proved in this context. This means that the semiring K must be known well enough to permit such decisions.

Keywords: algebra, systems, automata, languages, equations, the equality theorem.

Resumen

Esta investigación aborda la extensión de la teoría convencional de autómatas finitos al estudio del teorema de igualdad. Para ello, se propone un enfoque algebraico centrado en los conceptos de semianillos, K- subconjuntos reconocibles y K-Σ-autómatas. En este contexto, se demuestra la decidibilidad de cualquier par de K-subconjuntos reconocibles. Esto significa que el semianillo K debe conocerse con suficiente precisión para permitir tales decisiones.

Palabras clave: álgebra, sistemas, autómatas, lenguajes, ecuaciones, el teorema de igualdad.

1 Introduction

In systems theory, a class called Systems of Discrete Events (SED) is well known (see Branicky, 1995). It includes Manufacturing Systems, Chemical Systems, Economic Systems, Legal Systems, Air Traffic Systems, Telecommunications Systems; in short, any system whose states change in discrete time due to the occurrence of actions or events (see Caspi, 1991).

In this manuscript the Automata are presented by means of an algebraic approach as it is exposed in (Eilenberg, 1974), where the arguments and demonstrations are constructive; in this way it breaks with the conventional form imposed in the current literature on Automata.

It is of utmost importance to mention that the basic notions on which the theory of Automata is built are those of actions: events; and states: configurations of the system in time. Al-though these notions seem to be related to time, they are independent structurally speaking (see Mata, 2017). Indeed, at a logical level of abstraction, one is always interested, in the representation of a SED, only in the possible orders in which the actions of the system can occur (see Mata et al., 2018). This situation reasonably leads to verbally describe an SED as the set of all trajectories of a directed graph. Therefore, if Σ and Q are two sets representing actions and states respectively, and E is a proper subset of $Q\times\Sigma\times Q$, representing changes states by the occurrence of actions, then a SED is modeled by a quintuple $\mathbf{A}=(Q, \Sigma, E, I, T)$, where I and T are subsets of Q representing the states in which the system can start and the goals respectively. Finally, **A** is an automaton.

Now, from practice, we consider automata whose sets of actions and states are finite. Thus, the trajectories of an SED can be viewed as finite sequences of the form (q_0, α_1, q_1), (q_1, α_2, q_2),

..., (q_{n-1}, α_n, q_n), where each of these triples are elements of E. More precisely, the interest is focused on trajectories such that $q_0 \in I$ and $q_n \in T$. This set of trajectories corresponds to a set of labels of the form $\alpha_1\alpha_2, ..., \alpha_n$, which constitute the so-called behavior of the automaton A (or system dynamics), denoted by $|\mathbf{A}|$. This work consists of the study of mathematical structures (sets, functions, and relations) that can be described (or recognized) by finite state divides without auxiliary memory or storage capacity, and join it we see that the equality theorem silently assumes that a number of other facts are decidable.

2 Preliminaries

The main purpose of this work is to include the most relevant notions of automata theory: regular languages, operations with automata, among others, which allow to fix the terminology and notations that later lead to an extension problem.

Let Σ be a set. The free monoid Σ^* with basis Σ is defined as follows: the elements of Σ^* are n-tuples $s=(\alpha_1, \alpha_2, \cdots, \alpha_n)$, $n>0$, of elements of Σ. The integer n is called the length of s, which is denoted by $|s|$. If $w=(w_1, w_2, \cdots, w_m)$ is another element of Σ^*, then the product is defined by concatenation; that is, $sw=(\alpha_1, \alpha_2,\cdots, \alpha_n, w_1, w_2,\cdots, w_m)$. Then, one obtains the monoid Σ^* with unit $\theta = (\ )$, the 0-tuple. Clearly, $|sw|=|s|+|w|$ and $|\ \theta\ |=0$. Putting $\alpha=(\alpha)$, $\alpha\in\Sigma$, one can write $s=\alpha_1\alpha_2\cdots\alpha_n$, if $n>1$.

Any subset L of Σ^* is called a language over Σ. On the other hand, $s \in \Sigma^*$ is called a prefix of $w \in \Sigma^*$, denoted $s \preceq w$, if there exists a word $\sigma \in \Sigma^*$ such that $w=s\sigma$. Let $L \subset \Sigma^*$ be a language over Σ, the subset of all word prefixes of L is called the closure of L, denoted L; i.e., $\bar{L}=\{s \in \Sigma^*/\exists\ w \in \Sigma^*, sw \in L\}$. Finally, L is closed if $L=\bar{L}$.

For its part, automata theory is an approach that contains a state transition structure, which allows directing the analysis and synthesis by making use of the transition mechanism. Formally, let Σ be a finite alphabet. A Finite Automaton **A** (AF) over Σ or simply a deterministic Σ-automaton is a quin-tuple (Q, Σ, E ,I, T), where Q is a finite set whose elements are called states, I and T are subsets of Q called initial and final state sets respectively, and E is a subset of $Q\times\Sigma\times Q$, whose elements are called events. Additionally, if **A** has at most one initial state, and for all $q \in Q$ and $\alpha \in \Sigma$, there exists at most one event $(q, \alpha, p)\in E$, then **A** is called deterministic.

An event (q, σ, p) is denoted $q\xrightarrow{\sigma} p$, and this is said to begin at q and end at p with label σ.

A path c in A is a finite succession $c=(q_0, \alpha_1, q_1)(q_1, \alpha_2, q_2) \cdots (q_{k-1}, \alpha_k, q_k)$ of consecutive arcs, where q_0 and q_k are called the beginning and end of the path c respectively, and the integer $k\geq1$ is called the length of the path. The following notations are used for a path c: $q_0 \xrightarrow{\sigma_1} q_1 \rightarrow\cdots\xrightarrow{\sigma k} q_k$, $q_0 \xrightarrow{c} q_k$ or $c\colon q_0 \rightarrow q_k$. The element $=\alpha_1\alpha_2...\alpha_k \in \Sigma^*$ is called the label of c and is denoted by $|c|$. The length of s is denoted by $|s|$ and that of the path by $\|c\|$. Thus, it follows that $|s|=\|c\|=k$.

For each state q, we include the null path (trivial path) 1_q, which starts and ends at q. By definition, the null path has label θ and length 0; that is, $|1_q|= 0$ and $\|1_q\|=0$. Moreover, given two paths $c\colon p \rightarrow q$ and $c'\colon q \rightarrow r$, the path $cc'\colon p \rightarrow r$ (path composition) is defined by concatenation. then, $|cc'|=|c||c'|$ y $\|cc'\|=\|c\|+\|c'\|$.

Let $c\colon i \rightarrow t$ be a path in A, c is said to be a successful path if $I \in I$ and $t \in T$. The label of this path is called a successful label. The set of all successful labels in A is called the behavior or dynamics of A, and is denoted by $|A|$; i.e., $|A|=\{s \in \Sigma^*/\exists$ c: $i\rightarrow t$ in A, with $I \in I, t \in T, |c|=s\}$.

A language B of Σ^* is called regular if there exists a Σ-automaton A such that $B=|A|$.

In what follows we write $\alpha^*=\{\alpha\}^*$, for all $\alpha\in\Sigma$, in order to simplify the writing. Also, we treat $\alpha \in \Sigma$ and $s \in \Sigma^*$ as unitary sets.

Next, some basic automata operations are studied and their behaviors are analyzed. Let two AF be, $A=(Q_A, \Sigma, E_A, I_A, T_A)$ and $B=(Q_B, \Sigma, E_B, I_B, T_B)$, where $Q_A \cap Q_B=\emptyset$. The Σ-automaton union is given by $C=A\cup B= (Q_C, \Sigma, E_C, I_C, T_C)$, where $Q_C=Q_A\cup Q_B$, $I_C= I_A\cup I_B$, $T_C=T_A\cup T_B$. Moreover, an event is in E_C, if and only if, it is in E_A or it is in E_B. Therefore, a path is in C, if and only if, it is in A or it is in B.

The Σ-automaton product (or intersection) of A and B is given by $C=A\times B=(Q_C, \Sigma, E_C, I_c, T_C)$ where $Q_C=Q_A\times Q_B$, $I_C= I_A\times I_B$, $T_C=T_A\times T_B$. Consequently, an event $(p', p'')\xrightarrow{\alpha}(q', q'')$ is in E_C, if and only if, $p'\xrightarrow{\alpha} q'$ is an event in E_A and $p''\xrightarrow{\alpha} q''$ is an event in E_B.

On the other hand, we call the inverse automaton of **A** the Σ-automaton given by $A^\varphi=(Q, \Sigma, E^\varphi, I, T)$, where E^φ is the subset whose elements are the inverse events of E; that is, if $p \xrightarrow{\alpha} q$ is an event in E, then $q \xrightarrow{\alpha} p$ is an event in E^φ.

Note that, if c is a path in A with label $|c|=\alpha_1...\alpha_k$, then c^φ is a path in $\mathbf{A}^\varphi$ with label $|c^\varphi|= \varphi(\alpha_1...\alpha_k) = \alpha_k...\alpha_1$, where $\varphi : \Sigma^*\longrightarrow\Sigma^*$ is the inverse function defined by $\varphi(\theta)=\theta$, $\varphi(\alpha)=\alpha$, $\varphi(st)=\varphi(t)\varphi(s) =ts$.

It can be shown that the class of regular subsets is closed under union, intersection and inverse.

Now, some constructions on automata are given, relating them by a monoid homomorphism f: $\Gamma^*\rightarrow\Sigma^*$, where Γ and Σ are two alphabets. Also, f is assumed to be a fine homomorphism: f $(\alpha) \in \Sigma\cup\theta$, $\forall\ \alpha \in \Gamma$. The identities θ_1 and θ_2 of Γ^* and Σ^* respectively are referred to as θ.

Indeed, let $f : \Gamma^*\longrightarrow\Sigma^*$ be a fine homomorphism. We call the inverse image of A the Γ-automaton $f^{-1}(A)$, where Q, I, T are unperturbed, and the events are given by $p \overset{\gamma}{\rightarrow} q$, if $f(\gamma) = \alpha$ and $p \overset{\alpha}{\rightarrow} q$ is an event in A, and $q \overset{\gamma}{\rightarrow} p$, if $f(\gamma) = \theta$.

A path c': $p \rightarrow q$ in $f^{-1}(A)$ can be viewed as a pair (c, g), where c: $p \rightarrow q$ is a path in A and g $\in \Gamma^*$ is such that $|c'|$=g and f (g)=|c|; whence, it can be shown that if f : $\Gamma^*\longrightarrow\Sigma^*$ is a fine homomorphism and $A \subset \Sigma^*$ is regular, then $f^{-1}(A) \subset \Gamma^*$ is regular.

Finally, let f: $\Gamma^*\rightarrow\Sigma^*$ be a homomorphism such that $f(\gamma)\neq\theta$, for all $\gamma \in \Gamma$ (or equivalently that $f^{-1}(\theta)=\theta$ or still equivalently that $|g| \leqslant |f(g)|$, for all $g \in \Gamma^*$). Let $\mathbf{A} = (Q, \Gamma, E, I, T)$ be a Γ-automaton. We call the direct image of $\mathbf{A}$ the Σ-automaton $f(A)=(Q', \Sigma, E', I, T)$ where $Q' \supset Q$ and the events are determined as follows: let $\mathbf{p} \overset{\boldsymbol{\gamma}}{\rightarrow} \mathbf{q}$ be an event in A and $f(\gamma)=\alpha_1\cdots\alpha_n$, $n\geqslant1$; if n=1, then the arc $\mathbf{p} \overset{\boldsymbol{\alpha_1}}{\rightarrow} \mathbf{q}$ is in f (A); if n>1, then the arcs $\mathbf{p} \overset{\boldsymbol{\alpha_1}}{\rightarrow} \mathbf{q_1} \overset{\boldsymbol{\alpha_2}}{\rightarrow} \mathbf{q_2} \rightarrow\cdots\rightarrow \mathbf{q_{n-1}} \overset{\boldsymbol{\alpha_n}}{\rightarrow} \mathbf{q}$ are in f (**A**), where the n−1 intermediate states are new distinct states added to Q'. Repeating this, for every arc in A, f(A) is obtained.

As before, if f: $\Gamma^*\rightarrow\Sigma^*$ is a homomorphism such that $f^{-1}(\theta)=\theta$, and $A \subset \Gamma^*$ is regular, then $f(A) \subset \Sigma^*$ is regular.

3 K- Σ-Automatas

As a methodological support to formalize the technique describing the dynamics of an AF, the concept of multiplicity is included. This allows an extension grounded on mathematical objects (sets, functions, relations, among others) in the field of system dynamics.

To make it a little more precise, consider an AF **A**=(Q, Σ, E, I, T) with dynamics |**A**|. If c: i⟶t, I ∈ I, t ∈ T, |c|=s is in **A** and n determines the number of these paths, then one can define an application $\mu:\Sigma^*\rightarrow\mathbb{N}$ that specifies the multiplicity of the elements of Σ^*. This is referred to for $s \in \Sigma^*$, with multiplicity n. With abuse of language it is written μ=|**A**| and |**A**|(s)=n; whence, |**A**|(s)=0 expresses that s ∉ |**A**|. Finally, Σ^* is identified in what follows with |**A**|: $\Sigma^*\rightarrow\mathbb{N}$. It is also emphasized that any subset A of Σ^* is equivalent to a function $A:\Sigma^*\rightarrow\beta$, with β={0, 1}, in the sense that s s ∈ A ⇔ A(s)=1.

A fundamental structure for the development of this article is included. In fact, the notion of semiring is a weak structure of the conventional concept of ring.

Definition 1. A semiring K is a subset endowed with two operations: addition (+) and multiplication (.); such that (K,+) is a commutative monoid with neutral element 0 and (K, .) is a monoid with identity element 1. Moreover, for all x, y, z ∈ K one has that x(y + z)=xy + xz; (y + z)x=yx + zx; x0=0=0x. A semiring K is called commutative if (K, .) is commutative. Clearly, every ring with unity is a semiring.

Consider $\{x_i\}_{i\in I}$ an arbitrary collection of elements of a semiring K, with I a given set of indices.

Assuming finiteness of I, $\sum_{i\in I} x_i \in K$ **(1)**

The following properties with respect to the sum are true:

$I = \{i\} \Rightarrow \sum_{i\in I} x_i = x_i$ **(2)**

Let $I = \bigcup_{j\in J} I_j$ be a partition of I, z ∈ K⇒ $\sum_{i\in I} x_i = \sum_{j\in J} \left(\sum_{i\in I_j} x_i\right)$; $z(\sum_{i\in I} x_i) = \sum_{i\in I} zx_i$; $(\sum_{i\in I} x_i)z = \sum_{i\in I} x_i z$; $I = \emptyset \Rightarrow \sum_{i\in I} x_i = 0$. **(3)**

Considering **(1)** as the sum x+y and taking **(2), (3)**, (K,.), as axioms, we define $x_1 + x_2 := \sum_{i\in I} x_i$ with I={1,2}; and $0 := \sum_{i\in I} x_i$ if I=∅.

If I is finite, then clearly **(1)** is well defined. On the other hand, if I is an arbitrary index set, then **(1)** must be well-defined, and is an element of K. Thus, one has the concept of a complete semiring under the new definition. Consequently, every complete semiring is a semiring.

Definition 2. Given two semirings K and K′, a homomorphism φ:K⟶K′ is any function such that $\varphi(x_1+x_2) = \varphi(x_1)+\varphi(x_2)$, φ(0) = 0'; and $\varphi(x_1.x_2) = \varphi(x_1)\varphi(x_2)$, φ(1) = 1'.

Definition 3. A semiring K is called positive if it satisfies: 0≠1; if x + y = 0, then x = y = 0; if xy = 0, then x = 0 or y = 0.

K-subconjuncts, with K a semiring, are objects that allow identifying functions with their domains, and this constitutes a technical approach to notational handling and proof construction. In the following it is assumed that K is a nontrivial (0 ≠ 1) and commutative semiring.

Definition 4. Let X be a set. A K-subset A of X is any function A:X⟶K. For each x ∈ X, the element A(x) is called the multiplicity with which x belongs to A. If the values taken by A are 0 and 1, the K-subset A of X is said to be unambiguous.

Example 1. The subsets $\mathbf{X}, \emptyset$ and $\boldsymbol{x}$, for all $\mathbf{x} \in \mathbf{X}$, defined by $\mathbf{X(x) = 1}$, for all $\mathbf{x} \in \mathbf{X}$; $\emptyset(\mathbf{x}) = \mathbf{0}$, for all ; $\mathbf{x} \in \mathbf{X}$; $\mathbf{x(y)} = \begin{cases} \mathbf{1}, \ \mathbf{if\ x = y} \\ \mathbf{0}, \ \mathbf{in\ another\ case} \end{cases}$, respectively, are unambiguous.

The unambiguous subsets of x given in Example 1 are referred to as simplexes. If A is an unambiguous subset of X, then x∈A and A(x) = 1 indicate the same thing.

We define the sum or union operation as follows: for each family $\{A_i\}_{i\in I}$ of K-subsets of X, with I an arbitrary family of indices $\left(\bigcup_{\mathbf{i\in I}} \mathbf{A_j}\right)(\mathbf{x}) = \left(\sum_{\mathbf{i\in I}} \mathbf{A_i}\right)(\mathbf{x}) = \sum_{\mathbf{i\in I}} \mathbf{A_i(x)}$. **(4)**

Now, consider the operation product (or multiplication) of k ∈ K by a *K*-subset A as follows $(kA)(x) = kA(x)$ **(5)**

i.e., the result is a **K**-subset **kA**.

We have the following properties: $\mathbf{1A = A}, \mathbf{0A} = \emptyset, \mathbf{(k1k2)A = k1(k2A)}, \left(\sum_{\mathbf{i\in I}} \mathbf{k_i}\right)\mathbf{A} = \sum_{\mathbf{i\in I}} \mathbf{k_i A}$; and furthermore, $\mathbf{k}\left(\sum_{\mathbf{i\in I}} \mathbf{A_i}\right) = \sum_{\mathbf{i\in I}} \mathbf{k_i A}$.

We define the intersection A ∩ B of two K-subsets as (A ∩ B)(x) = A(x)B(x).

The sum $\sum_{\mathbf{x\in X}} \mathbf{A(x)x}$ is called the expansion of A. This expression is useful for manipulating K-subsets.

Example 2. $\mathbf{kA} = \sum_{\mathbf{x\in X}} \mathbf{kA(x)x}$; and further, $\mathbf{A \cap B} = \sum_{\mathbf{x\in X}} \mathbf{A(x)B(x)x}$.

The study of the multiplication or product of K-subsets of S is included, being (S, ·) a semigroup. Indeed, a category of objects with the above-mentioned operations of addition and multiplication is included to formalize matrix notions and structures.

Definition 5. Let (S, ·) be a semigroup, A and B K-subsets of S, where K is a complete semiring. The K-subset product AB is given by $\mathbf{(AB)(z)} = \sum_{\mathbf{xy=z}} \mathbf{A(x)B(y)}$.

It is clear that the operation AB is associative. Therefore, K^M is a semiring with identity θ, provided M is a monoid, where θ is the identity of M.

Consider P, Q two finite sets. A K-subset of P × Q is any matrix whose rows and columns are indexed using the elements of P and Q respectively, and whose entries are in K. Then, $A \in K^{P\times Q}$ is written A_{pq} instead of A(p,q); thus, the matrix is written $A = [A_{pq}]$.

The matrix sum operation is established by the sum of K-subsets. That is, $(A+B)_{pq} = A_{pq} + B_{pq}$, provided that $B \in K^{P\times Q}$.

The matrix multiplication operation is given as follows: let $A \in K^{P\times Q}$ and $B \in K^{Q\times R}$ be, then $\mathbf{(AB)_{pr}} = \sum_{\mathbf{q\in Q}} \mathbf{A_{pq}B_{qr}}$.

Some properties of multiplication: if P = Q, then $K^{P\times P}$ is a semiring with unit 1_p, where $(\mathbf{1_p})_{\mathbf{q}'} = \begin{cases} \mathbf{1}, \ \mathbf{si\ q = q'} \\ \mathbf{0}, \ \mathbf{si\ q \neq q'} \end{cases}$, A is called a row vector, provided that $A \in K^{P\times Q}$ and P is unitary. Also, A is called a column vector, provided that Q is unitary.

Definition 6. Let Σ be a finite alphabet and K a commutative semiring. A K-Σ-automaton **A** (or a deterministic K-Σ-automaton) is a quintuple, **A**=(Q, Σ, E, I, T) where Q is a finite set, I and T are K-subsets of Q and E is K-subsets of Q×Σ×Q.

Given **A**=(Q, Σ, E, I, T) a K-Σ-automaton, if E(p,α,q) = k≠ 0, then there is said to be an arc from p to q denoted $\mathbf{p} \overset{\mathbf{k\alpha}}{\to} \mathbf{q}$, labeled kα. Also, $\mathbf{p} \overset{\mathbf{k\alpha}}{\to} \mathbf{q}$ is said to be in **A**.

Thus, analogous to Σ-automata, one can consider paths $\mathbf{c}: \mathbf{p} \to \mathbf{q}$. Then, if c is a path $\mathbf{p} \overset{\mathbf{k_1\alpha_1}}{\to} \mathbf{q_1} \overset{\mathbf{k_2\alpha_2}}{\to} \cdots \mathbf{q_{n-1}} \overset{\mathbf{k_n\alpha_n}}{\to} \mathbf{q}$, then |c|=ks is its label, with $k = k_1 \cdot\ \cdot \cdot k_n$ and $s = \alpha_1 \cdots \alpha_n$, and length ‖c‖=n=|s|.

Definition 7. Let **A**=(Q, Σ, E, I, T) be a K-Σ-automaton. The behavior or dynamics of **A** is a K-subset of Σ*, denoted |**A**|, and is given by $|\mathbf{A}| = \sum_{\mathbf{p,q\in Q}} \sum_{\mathbf{c}} \mathbf{I(p)\,|c|T(q)}$, with c varying over all paths $\mathbf{c}: \mathbf{p} \to \mathbf{q}$; i.e., $|\mathbf{A}|(\mathbf{s}) = \sum_{\mathbf{p,q\in Q}} \sum_{\mathbf{k\in c}} \mathbf{I(p)\,kT(q)}$, where C = {k∈K: ∃ c:p→q, |c| = ks}.

Note that the K-subset E of Q×Σ×Q can be viewed as a function E:Q×Σ×Q→K. In what follows we write $E(p, \alpha, q) = E_{pq}(\alpha)$. Then, for all p, q∈Q, one has that E_{pq} is a K-subset of Σ. Then, E can be identified with a matrix E: Q×Q→K^Σ called the transition matrix.

Any K-subset of Σ can be extended to a K-subset of Σ* as follows: for p, q ∈ Q, E_{pq}: Σ*⟶K, $\mathbf{E_{pq}(s)} =$

$\begin{cases}\mathbf{E_{pq}(s)}, & \mathbf{si\ s \in \Sigma}\\ \mathbf{0}, & \mathbf{si\ s \notin \Sigma}\end{cases}$. Thus, E can be viewed as a K^{Σ^*}-subset of Q × Q; that is, a Q × Q matrix with entries in K^{Σ^*}. Consequently, since K^{Σ^*} is a semiring, the corresponding operations are used.

Now, for each n∈ℕ, we consider the matrices E^n: $Q\times Q \rightarrow K^{\Sigma^*}$ by $E^0=1_Q$, $E^1=E$ and $E^n=EE^{n-1}$, n≥2, with $E^n{}_{pr} = \sum_{r\in Q} E_{pr}E^{n-1}{}_{rq}$, where p, q∈Q. It is clear that if s∈Σ* and |s|≠n, one has that $E^n{}_{pq}(s) = 0$, with p, q∈Q; whereupon, $\{E^n{}_{pq}\}_{n\in\mathbb{N}}$ is locally finite. Then, one can define $E^*{}_{pq} = \sum_{n=0}^{\infty} E^n{}_{pq}$, and hence, it results in the matrix $E^*: Q\times Q \rightarrow K^{\Sigma^*}$, $E^*=1_Q+E+E^2+\cdots+E^n+\cdots$ called the extended transition matrix.

For each s ∈ Σ*, let $E^*(s)=E^*_{pq}(s)\in K^{Q\times Q}$ be, if $s=\alpha_1\cdots\alpha_n$, it follows that $E^*(s)=E^n(s)=E(\alpha_1)\cdots E(\alpha_n)=E^*(\alpha_1)\cdots E^*(\alpha_n)$.

Theorem 1. For any p,q∈Q, the K-subset $\boldsymbol{E^*{}_{pq}}$ is the sum of all labels of $\mathbf{c}: \mathbf{p} \rightarrow \mathbf{q}$ in A.

Proof: Let p, q ∈ Q be, as $\mathbf{E^*{}_{pq}} = \sum_{\mathbf{pq=1}}^{\infty} \mathbf{E^n{}_{pq}}$, it suffices to show that $\mathbf{E^n{}_{pq}}$ is the sum of all path labels of length n. If n = 0, then $\mathbf{E^0{}_{pq}} = \begin{cases}\boldsymbol{\theta}, & \mathbf{si\ p = q}\\ \mathbf{0}, & \mathbf{si\ p \neq q}\end{cases}$, where $\boldsymbol{\theta}$ is the identity of K^{Σ^*}. If n=1, then $\mathbf{E^1{}_{pq}} = \sum_{\mathbf{r\in Q}} \mathbf{E_{pq}(1_Q)_{rp}} = \sum_{\mathbf{r\in Q}} \mathbf{E_{pr}E^0{}_{rq}}$. Assume that the result is true for n−1, n ≥2; i.e., $\mathbf{E^{n-1}{}_{pq}} = \sum_{\mathbf{r_1,r_2,\ldots,r_{n-1},\in Q}} \mathbf{E_{pr_1}E_{pr_1r_2}}\cdots \mathbf{E_{r_{n-1}}} = \sum_{\mathbf{r\in Q}} \mathbf{E_{pr}E^{n-2}{}_{rq}}$, then $\mathbf{E^n{}_{pq}} = \sum_{\mathbf{r\in Q}} \mathbf{E_{pr}E^{n-1}{}_{rq}} = \sum_{\mathbf{r_1,\in Q}} \mathbf{E_{pr_1}} \left(\sum_{\mathbf{r_1,r_2,\ldots,r_n,\in Q}} \mathbf{E_{r_1r_2}E_{r_1r_3}}\cdots \mathbf{E_{r_nq}}\right) = \sum_{\mathbf{r_1,r_2,\ldots,r_n,\in Q}} \mathbf{E_{pr_1r_2}E_{r_1r_2}}\cdots \mathbf{E_{r_nq}} = \sum_{\mathbf{r\in Q}} \mathbf{E_{pr}E^{n-2}{}_{rq}}$. Consequently, $\mathbf{E^n{}_{pq}}$ is the sum of the labels of paths with length n. Therefore, $\boldsymbol{E^*{}_{pq}}$ is the sum of the labels of $\mathbf{c}: \mathbf{p} \rightarrow \mathbf{q}$ in A.

Corollary 1. The behavior of A is $|A| = \mathbf{I}\mathbf{E}^*\mathrm{T}$ with I viewed as a row vector and T as a column vector.

Proof: $|\mathbf{A}| = \sum_{\mathbf{p,q\in Q}} \sum_{\mathbf{c}} \mathbf{I(p)\,|c|\,T(q)} = \sum_{\mathbf{p,q\in Q}} \mathbf{I(p)\,E^*{}_{pq}\,T}$.

Definition 8. Let K be a commutative semiring, and Σ be a finite alphabet. A K-subset A of Σ* is called regular, if there exists a K-Σ-automaton **A** such that |**A**| = A.

In what follows, it is always assumed that given two K-Σ-automata $\mathbf{A}=(Q_A, \Sigma, E_A, I_A, T_A)$ and $\mathbf{B}=(Q_B, \Sigma, E_B, I_B, T_B)$, $Q_A\cap Q_B=\emptyset$.

Definition 9. Let A, B be two K-Σ-automata, the K-Σ-automata union of A and B is given by $A\cup B=(Q_{A\cup B},\Sigma,E_{A\cup B},I_{A\cup B},T_{A\cup B})$, where, $Q_{A\cup B} = Q_A\cup Q_B$,

$$\mathbf{I_{A\cup B}(p)} = \begin{cases}\mathbf{I_A(p)}, & \mathbf{si\ p \in Q_A}\\ \mathbf{I_B(p)}, & \mathbf{si\ p \in Q_B}\end{cases},$$

$$\mathbf{T_{A\cup B}(p)} = \begin{cases}\boldsymbol{T}_\mathbf{A}\mathbf{(p)}, & \mathbf{si\ p \in Q_A}\\ \mathbf{T_B(p)}, & \mathbf{si\ p \in Q_B}\end{cases},$$

y

$$\mathbf{E_{A\cup B}(p,\alpha,q)} = \begin{cases}\mathbf{E_A(p,\alpha,q)}, & \mathbf{si\ p,q \in Q_A}\\ \mathbf{E_B(p,\alpha,q)}, & \mathbf{si\ p,q \in Q_B}\\ \mathbf{0}, & \mathbf{en\ otro\ caso}\end{cases}$$

Proposition 1. The union of two regular K-subsets of Σ* is a regular K-subset of Σ*.

Proof: Consider A and B two regular K-subsets of Σ*, and A, B two K-Σ-automata such that |A|=A and |B|=B. Let A ∪ B be a K-Σ-automaton. Then, for all s∈Σ*,

$$|\mathbf{A} \cup \mathbf{B}| = \left(\sum_{\mathbf{p,q\in Q_{A\cup B}}} \sum_{\mathbf{c}} \mathbf{I_{A\cup B}(p)\,|c|\,T_{A\cup B}(q)}\right)\mathbf{(s)}$$

$$= \sum_{\mathbf{p,q\in Q_{A\cup Q_B}}} \sum_{\mathbf{k}} \mathbf{I_{A\cup B}(p)\,k\,T_{A\cup B}(q)},$$

where k∈K is such that there exists $\mathbf{c}: \mathbf{p} \rightarrow \mathbf{q}$ in A∪B with |c|=ks; thus,

$$\sum_{\mathbf{p,q\in Q_{A\cup Q_B}}} \sum_{\mathbf{k}} \mathbf{I_{A\cup B}(p)\,k\,T_{A\cup B}(q)} = \sum_{\mathbf{p,q\in Q_A}} \sum_{\mathbf{k}} \mathbf{I_A(p)\,k\,T_A(q)} + \sum_{\mathbf{p,q\in Q_B}} \sum_{\mathbf{k}} \mathbf{I_B(p)\,k\,T_B(q)} = |\mathbf{A}|\mathbf{(s)} + |\mathbf{B}|\mathbf{(s)}$$

$$= \mathbf{A(s)} + \mathbf{B(s)}$$

$$= (\mathbf{A} \cup \mathbf{B})\mathbf{(s)}.\ \mathbf{Asi},\ |\mathbf{A} \cup \mathbf{B}| = \mathbf{A} \cup \mathbf{B}.$$

Definition 10. Let A, B be two K-Σ-automata. The K-Σ-automaton product (or intersection) of A and B is given by $A\times B=(Q_{A\times B},\Sigma,E_{A\times B},\mathbf{I}_{A\times B},T_{A\times B})$, with $Q_{A\times B}=Q_A\times Q_B$, $I_{A\times B}((p,q))=\mathbf{I}_A(p)\mathbf{I}_B(q)$, $T_{A\times B}((p,q))=T_A(p)T_B(q)$,

$E_{A\times B}((p,q),\alpha,(p',q'))=E_A(p,\alpha,p')E_B(q,\alpha,q')$.

Proposition 2. The intersection of two K-regular subsets of Σ^* is a K-regular subset of Σ^*.

Proof: Let A and B be two regular K-subsets of Σ^* and A, B be two K-Σ-automata such that $||A|=A$ and $|B|=B$. Let A×B be the K-Σ-automaton. Then, for all $s\in\Sigma^*$,

$$|\mathbf{A}\times\mathbf{B}|(s)=\left(\sum_{(\mathbf{p},\mathbf{q}),(\mathbf{p}',\mathbf{q}')\in Q_{A\times}Q_B}\sum_{c}\mathbf{I}_{A\times B}(\mathbf{p},\mathbf{q})|\mathbf{c}|\mathbf{T}_{A\times B}(\mathbf{p}',\mathbf{q}')\right)(s)$$
$$=\left(\sum_{p,p',\in Q_A;q,q',\in Q_B}\sum_{c=(c',c'')}\mathbf{I}_A(\mathbf{p})\mathbf{I}_B(q)|(\mathbf{c}',\mathbf{c}'')|\mathbf{T}_A(\mathbf{p}')\mathbf{T}_B(\mathbf{q}')\right)(s)$$
$$=\sum_{p,p',\in Q_A;q,q',\in Q_B}\sum_{k_1,k_2}\mathbf{I}_A(\mathbf{p})\mathbf{k_1}\mathbf{T}_A(\mathbf{p}')\mathbf{I}_B(q)\mathbf{k_2}\mathbf{T}_B(\mathbf{q}')$$
$$=\sum_{p,p',\in Q_A}\sum_{k_1,}I_A(p)k_1T_A(p')\sum_{q,q',\in Q_B}\sum_{k_2}I_B(q)k_2T_B(q')$$
$$=\left(\sum_{p,p',\in Q_A}\sum_{c'}I_A(p)|c'|\,T_A(p')\right)(s)\left(\sum_{q,q',\in Q_B}\sum_{c'}I_B(q)|c''|T_B(q')\right)(s)$$
$$=|A|(s)|B|(s)=(\,|A|\,\cap\,|B|)(s)=(\,A\,\cap\,B)(s).$$

Where $\mathbf{c}':\mathbf{p}\to\mathbf{p}'$ is a path in A, $\mathbf{c}'':\mathbf{q}\to\mathbf{q}'$ is a path in B, $|c'|=k_1s$, $|c''|=k_2s$ and $k_1k_2=k$ with $ks=|c|$.

Definition 11. Let A=(Q,Σ,E,I,T) be a K-Σ-automaton. We call K-Σ-automaton inverse K-Σ-automaton $A^\varphi=(Q,\Sigma,E_\varphi,I_\varphi,T_\varphi)$, where $I_\varphi(q)=T(q)$, $T_\varphi(q)=I(q)$, and $E_\varphi(p,\alpha,q)=E(p,\alpha,q)$.

Remark 1. A path c^φ: p⟶q in A^φ, with label $|c^\varphi|=ks$, is given by a path c: q⟶p in A with label $|c|=k\varphi(s)$, where φ: $\Sigma^*\to\Sigma^*$ is the inverse function defined by $\varphi(\theta)=\theta$, $\varphi(\alpha)=\alpha$, $\varphi(st)=\varphi(t)\varphi(s)$, where $t,s\in\Sigma^*$.

Proposition 3. If A is a regular K-subset of Σ^* and φ: $\Sigma^*\to\Sigma^*$ is the inverse function given in Remark 1, then $\mathbf{A}\circ\boldsymbol{\varphi}$ is a regular K-subset of Σ^*; that is, the class of regular K-subsets of Σ^* is stable under inverse function.

Proof: Let A be a K-regular subset of Σ^* and A be a K-Σ-automaton such that |A|=A. Consider A^φ the inverse K-Σ-automaton of A, then for all $s\in\Sigma^*$,

$$|\mathbf{A}^{\boldsymbol{\varphi}}|(\mathbf{s})=\left(\sum\nolimits_{\mathbf{p},\mathbf{q}\in\mathbf{Q}}\sum\nolimits_{\mathbf{c}^{\boldsymbol{\varphi}}}\mathbf{I}_{\boldsymbol{\varphi}}(\mathbf{p})\,|\mathbf{c}^{\boldsymbol{\varphi}}|\mathbf{T}_{\boldsymbol{\varphi}}(\mathbf{q})\right)(\mathbf{s})$$
$$=\sum_{\mathbf{p},\mathbf{q}\in\mathbf{Q}}\sum_{\mathbf{k}}\mathbf{T}(\mathbf{p})\,\mathbf{kI}(\mathbf{q}),$$

with $|c^\varphi|=ks$, where $k\in K$ and

$$|\mathbf{c}|=\mathbf{k}\boldsymbol{\varphi}(\mathbf{s})=\left(\sum_{\mathbf{p},\mathbf{q}\in\mathbf{Q}}\sum_{c}\mathbf{I}\,(\mathbf{q})\,|\mathbf{c}\,|\mathbf{T}\,(\mathbf{p})\right)(\boldsymbol{\varphi}(\mathbf{s}))$$
$$=(|\mathbf{A}\,|)(\boldsymbol{\varphi}(\mathbf{s}))=(\mathbf{A}\circ\boldsymbol{\varphi})(\mathbf{s}).$$

Thus, $|\mathbf{A}^{\boldsymbol{\varphi}}|=\mathbf{A}\circ\boldsymbol{\varphi}$.

Definition 12. Let A=(Q, Σ, E, I, T) be a K-Σ-automaton and f: $\Gamma^*\to\Sigma^*$ is a fine homomorphism. We call the inverse image of A the K-Γ-automaton $f^{-1}(A)=(Q,\Gamma,E',I,T)$, where E' is given by

$$\mathbf{E}'\,(\mathbf{p},\boldsymbol{\gamma},\mathbf{q})=\begin{cases}\mathbf{E}\,(\mathbf{p},\boldsymbol{\alpha},\mathbf{q}), & \mathbf{si}\;\;\mathbf{f}(\boldsymbol{\gamma})=\boldsymbol{\alpha}\;\;\mathbf{y}\;\;\mathbf{E}\,(\mathbf{p},\boldsymbol{\alpha},\mathbf{q})\neq\mathbf{0}\\ \mathbf{1}, & \mathbf{si}\;\;\mathbf{f}(\boldsymbol{\gamma})=\boldsymbol{\theta}\;\;\mathbf{y}\;\;\mathbf{p}=\mathbf{q}\\ \mathbf{0}, & \mathbf{otherwise}.\end{cases}$$

Remark 2. A path $\mathbf{c}:\mathbf{p}\to\mathbf{q}$ in $f^{-1}(A)$, with label |c| = kg, is given as c=(c',g) where $\mathbf{c}':\mathbf{q}\to\mathbf{q}$ is a path in A and $g\in\Gamma^*$ is such that |c'| = kf(g); that is, a path $\mathbf{c}:\mathbf{p}\to\mathbf{q}$ in $f^{-1}(A)$, which associates some $g\in\Gamma^*$ in its label, is given by a path c' in **A** which associates f(g) in its label.

Proposition 4. If f: $\Gamma^*\to\Sigma^*$ is a fine homomorphism and A is K-regular subset of Σ^*, then A∘f is a K-regular subset of Γ^*.

Proof: Let **A** be a K-Σ-automaton such that |**A**|=A. Consider the K-Γ-automaton $f^{-1}(\mathbf{A})$, the inverse image of **A**, and let $g\in\Gamma^*$ be, then

$$|\mathbf{f}^{\,-1}(\mathbf{A})|(\mathbf{g})=\left(\sum_{\mathbf{p},\mathbf{q}\in\mathbf{Q}}\sum_{c}\mathbf{I}\,(\mathbf{p})\,|\mathbf{c}\,|\mathbf{T}\,(\mathbf{q})\right)(\mathbf{g})$$
$$=\left(\sum_{\mathbf{p},\mathbf{q}\in\mathbf{Q}}\sum_{\mathbf{c}'}\mathbf{I}\,(\mathbf{p})\,|\mathbf{c}\,'|\mathbf{T}\,(\mathbf{q})\right)(\mathbf{f}(\mathbf{g}))=|\mathbf{A}|(\mathbf{f}(\mathbf{g}))$$
$$=(\mathbf{A}\circ\mathbf{f})(\mathbf{g}).$$

Thus, $|\mathbf{f}^{\,-1}(\mathbf{A})|=\mathbf{A}\circ\mathbf{f}$.

Definition 13. Let f: $\Gamma^*\to\Sigma^*$ be a homomorphism, such that $f(\gamma)\neq\theta$, $\forall\gamma\in\Gamma$, and let A=(Q,Γ,E,I,T) be a K-Γ automaton. We call the direct image of A the K-Σ-automaton f(A)=(Q',Σ,E',I',T'), where Q'⊃Q and E' are given as follows: Let E(p,γ,q) = k≠0 in A, and let $f(\gamma)=\alpha_1\alpha_2\cdots\alpha_n$, with n≥1. If n=1, then $E'(p,\alpha_1,q)=E(p,\gamma,q)=k$, is an arc in f (A). If n>1, consecutive arcs $\mathbf{p}\overset{\mathbf{k_1\alpha_1}}{\to}\mathbf{q_1}\overset{\mathbf{k_2\alpha_2}}{\to}\cdots\mathbf{q_{n-1}}\overset{\mathbf{k_n\alpha_n}}{\to}\mathbf{q}$ with $k_1\cdots k_n=k$, are in f(A), where $q_1,\cdots,q_{n-1}$ are new states added to Q. These determine states of Q'. Finally, repeating this process for every arc in A, we obtain f(A), with

$$\mathbf{T}\,(\mathbf{p}')=\begin{cases}\mathbf{I}\,(\mathbf{p}'), & \mathbf{si}\;\;\mathbf{p}'\in\mathbf{Q}\\ \mathbf{0}, & \mathbf{si}\;\;\mathbf{p}'\in\mathbf{Q}'\backslash\mathbf{Q}\end{cases}$$

y

$$\mathbf{T}'\,(\mathbf{p}')=\begin{cases}\mathbf{T}\,(\mathbf{p}'), & \mathbf{si}\;\mathbf{p}'\in\mathbf{Q}\\ \mathbf{0}, & \mathbf{si}\;\mathbf{p}'\in\mathbf{Q}'\backslash\mathbf{Q}\end{cases}$$

Remark 3. To say that f: $\Gamma^*\to\Sigma^*$ is a homomorphism such that $f(\gamma)\neq\theta$, $\forall\gamma\in\Gamma^*$ is equivalent to saying that $f^{-1}(\theta)=\theta$, or it is also equivalent to saying that $f^{-1}(s)$ is finite, for all $s\in\Sigma^*$.

Proposition 5. Let f: $\Gamma^*\to\Sigma^*$ be a homomorphism such that $f^{-1}(s)$ is finite, for all $s\in\Sigma^*$. If A is K-regular subset of Γ^*, then $f(A)$: $\Sigma^*\to K$, given by

$$\mathbf{f(A)\ (s)} = \sum_{\mathbf{g}\in\mathbf{f}^{-1}(s)} \mathbf{A(g)}$$

is a K-regular subset of Σ^*.

Proof: Consider $\mathbf{A}$ a K-Γ-automaton such that $|\mathbf{A}|$=A. Let f(A) be the K-Σ-automaton direct image of $\mathbf{A}$, then, for all $s\in\Sigma^*$, it follows that

$$|\mathbf{f(A)}| = \sum_{\mathbf{p',q'\in Q'}}\sum_{c'} \mathbf{I'(p')\,|c'|\,T'(q')}$$

$$= \left(\sum_{\mathbf{p,q\in Q}}\sum_{\mathbf{c'}} \mathbf{I(p)\,|c'|\,T(q)}\right)$$

Then, |

$$|\mathbf{f\ (A)}| = \sum_{\mathbf{p,q\in Q}}\sum_{k} \mathbf{I(p)\,kT(q)},$$

with k varying over $\mathbf{c'}$: $\mathbf{p}\to\mathbf{q}$, with |c'|=ks, s=f(g), kg=|c|, $\mathbf{c}$: $\mathbf{p}\to\mathbf{q}$ in A. Thus,

$$|\mathbf{f\ (A)}| = \sum_{\mathbf{p,q\in Q}}\sum_{k} \mathbf{I(p)\,kT(q)},$$

with k varying over all paths $\mathbf{c}$: $\mathbf{p}\to\mathbf{q}$, such that |c'|=kf(g), $g\in f^{-1}(s)$, kg = |c|, c: p → q in A. Therefore,

$$|\mathbf{f\ (A)}|(\mathbf{s}) = \sum_{\mathbf{g}\in\mathbf{f}^{-1}(s)}\left(\sum_{\mathbf{p,q\in Q}}\sum_{k} \mathbf{I(p)\,kT(q)}\right)$$

with k varying over all paths $\mathbf{c}$: $\mathbf{p}\to\mathbf{q}$ in A, kg = |c|; that is,

$$|\mathbf{f\ (A)}|(\mathbf{s}) = \sum_{\mathbf{g}\in\mathbf{f}^{-1}(s)} |\mathbf{A}|(\mathbf{g}) = \sum_{\mathbf{g}\in\mathbf{f}^{-1}(s)} \mathbf{A(g)}$$

Finally, f(A) is a K-regular subset of Σ^*.

Definition 14. A K-Σ-automaton A=(Q,Σ,E,I,T) is called normalized if $\mathbf{I}=\mathbf{i}$ and T= t are two distinct simple $\mathbf{K}$-subsets, and there exist no arcs $\mathbf{p}\xrightarrow{\mathbf{k\alpha}}\mathbf{i}$, $t\xrightarrow{\mathbf{k\alpha}}\mathbf{q}$ with k≠0; that is, $\mathbf{E}(q,\alpha,\mathbf{i})=\mathbf{E}(t,\alpha,q)=0$, for all q∈Q and $\alpha\in\mathbf{\Sigma}$.

Remark 4. If A is normalized, then $|A|(\theta)=0$. Then, if one observes Σ^+ as a B-subset of Σ^*, one obtains $|A|\subset\Sigma^+$.

Proposition 6. For every K-Σ-automaton A there exists a normalized K-Σ-automaton A' such that $|A'| = |A| \cap \Sigma^*$.

Proof: Let $\mathbf{A}$=(Q,Σ,E,I,T) be a K-Σ-automaton, and let Q'=Q ∪ i ∪ t, where i and t are two different new states. Consider the new matrix $\mathbf{E}'$ as follows: $\mathbf{E'_{pq}} = \mathbf{E_{pq}}$,

$\mathbf{E'_{iq}} = \sum_{\mathbf{p\in Q}} \mathbf{I_p E_{pq}}$, $\mathbf{E'_{pt}} = \sum_{\mathbf{q\in Q}} \mathbf{E_{pq} T_q}$, $\mathbf{E'_{it}} = \sum_{\mathbf{p,q\in Q}} \mathbf{I_p E_{pq} T_q}$,

$\mathbf{E'_{pi}} = \mathbf{E_{ti}} = \mathbf{E_{tq}} = \emptyset$, where $\mathbf{I_p} = \mathbf{I(p)}$ and $\mathbf{T}_q = \mathbf{T(q)}$. A calculation determines that $\mathbf{E'}^*_{it} = \mathbf{IE^+T}$, where $\mathbf{E^+} = \mathbf{E} + \mathbf{E^2} +\cdots+ \mathbf{E}^n +\cdots= \mathbf{EE^+}$.

The K-Σ-automaton A'=(Q',Σ,E',i, t) is normalized, and using **Corollary 1** one has $|\mathbf{A'}| = \mathbf{E'}^*_{it} = \mathbf{IE^+T} = \mathbf{IE^*T} \cap \mathbf{\Sigma^+} = |\mathbf{A}| \cap \mathbf{\Sigma^+}$.

Remark 5. The construction of A' from A, as in Proposition 6, is always interpreted as the normalization design.

Proposition 7. Let A be a K-regular subset of Σ^*, and let k∈K, then kA is a K-regular subset of Σ^+.

Proof: Let A be a K-Σ-automaton such that |A| = A, and let k∈K be, consider the K-Σ-automaton kA=(Q,Σ,E,kI,T), where $(kI)_q=kI_q$, then

$$|\mathbf{kA}| = \sum_{\mathbf{p,q\in Q}} \mathbf{kI_p E^*_{pq}}\, \boldsymbol{T_q} = \mathbf{k}\sum_{\mathbf{p,q\in Q}} \mathbf{I_p E^*_{pq}}\, \boldsymbol{T_q}$$
$$= \mathbf{k|A|} = \mathbf{kA}$$

Proposition 8. A K-subset A of Σ^* is regular, if and only if, the K-subset A'=A ∩ Σ^+ is also regular.

Proof: Let A be a K-Σ-automaton such that |A| = A. Then, there exists a normalized K-Σ-automaton A' such that $|A'|=|A|\cap\Sigma^+=A\cap\Sigma^+=A'$. Reciprocally, suppose that A $\cap\mathbf{\Sigma^+}=\mathbf{A'}$ is a $\mathbf{K}$-regular subset of $\mathbf{\Sigma}^*$. Since $\mathbf{A'}(\theta)=0(\mathbf{\Sigma^+}(\theta)=0)$, it follows that A=kθ+A', where k=A(θ). Then, since θ is a K-regular subset of Σ^*, then from Proposition 1 and Proposition 7 one has that A is regular.

Proposition 9. If A and B are two K-regular subsets of Σ^*, then AB is a K-regular subset.

Proof: Let A=kθ+A',B=lθ+B', with A'=A∩Σ^+, B'=B∩Σ^+, k=A(θ) and l=B(θ). Then, AB=kl(θ)+kB'+lA'+A'B'. It suffices to show that A'B' is regular, for this, let $A=(Q_1,\Sigma,E_1,i_1,t_1)$ and $B=(Q_2,\Sigma,E_2,i_2, t_2)$ be two K-Σ-normalized automata recognizing A' and B' respectively. Consider the normalized K-Σ-automaton $C=(Q,\Sigma,E,i_1,t_2)$, where Q is a partition of Q_1 and Q_2, except when $t_1=i_2$. Then, an arc in C is either an arc in A or is an arc in B. Thus, $|\mathbf{C}| = \mathbf{E}^*_{\mathbf{i_1t_2}} = \mathbf{E}^*_{\mathbf{1i_1t_2}} = \mathbf{E}^*_{\mathbf{2i_2t_2}} = |\mathbf{A}|\,|\mathbf{B}| = \mathbf{A'B'}$.

Proposition 10. Let A be a K-regular subset of Σ^+. Then, the K-subsets $A^+=A+A^2+\cdots+A^n+\cdots$ and

$A^*=\theta+A+A^2+\cdots+A^n+\cdots$ are regular.

Proof: Since $A\subset\Sigma^+$, it follows that $A^n(s)=0$, when $|s|<n$. Then, $\{A^n\}_{n\in N}$ is locally finite; thus, A^+ and A^* are well defined. Let **A**=(Q,Σ,E,i,t) be a K-Σ-normalized automaton recognizing A. Then, considering i and t as a single state, which is initial and final, we obtain a K-Σ-normalized automaton A^*, where $|A^*|=A^*$. Thus, A^* is regular. Then, since $A^+=A^*\cap\Sigma^+$, it follows that A^+ is regular.

Theorem 2. A K-subset A of Σ^+ is regular, if and only if, there exists an integer n>1 and a matrix E n×n whose entries are K-subsets of Σ such that $A=E^+$.

Proof: Let $\boldsymbol{A}$ be a $\boldsymbol{K}$-regular subset of $\boldsymbol{\Sigma}+$, and $\boldsymbol{A}=(\boldsymbol{Q},\boldsymbol{\Sigma},\boldsymbol{E},\boldsymbol{i},\boldsymbol{t})$ be a $\boldsymbol{K}-\boldsymbol{\Sigma}-$normalized automaton that recognizes $\boldsymbol{A}$. There is no loss of generality in assuming $\boldsymbol{Q}=\boldsymbol{1},\cdots,\boldsymbol{n}\}$, with $\boldsymbol{i}=\boldsymbol{1}$ and $\boldsymbol{t}=\boldsymbol{n}$. Since $\boldsymbol{i}\neq\boldsymbol{t}$ has to $\boldsymbol{n}>\boldsymbol{1}$. Then, from Corollary 1 it follows that $|\mathbf{A}|=\mathbf{E}^*{}_{\boldsymbol{in}}=\mathbf{E}^+{}_{\boldsymbol{in}}=\mathbf{A}$. Reciprocally, if, $\mathbf{A}=\mathbf{E}^+{}_{\boldsymbol{1n}}$, where $\boldsymbol{E}$ is a matrix n×n of K-subsets of Σ and n>1, then, with Q={1,⋯,n}, I=1 and T=n, one obtains a K-Σ-automaton A=(Q,Σ,E,1,n) such that $|A|=E^+=A$.

Corollary 2. If E is a matrix n×n of K-subsets of Σ, then for any $1\le i,j\le n$, the K-subsets $\mathbf{E}^*{}_{\boldsymbol{ij}}$ and $\mathbf{E}^+{}_{\boldsymbol{ij}}$ are regular.

Proof: It follows from Theorem 2.

Proposition 11. Let φ: K→K′ be a homomorphism of semirings. If A is a regular K-subsubset of Σ^*, then φ ∘ A is a regular K′-subset of Σ^*.

Proof: Let A=(Q,Σ,E,I,T) be a K-Σ-automaton such that |A|=A, then the K′-Σ-automaton φ(A)=(Q,Σ,φ(E),φ(I), φ(T)) satisfies |φ(A)|=φ(|A|) =φ(A). Since K is a positive semiring, then T: K→B, given by T (0) =0 and T (x)=1, for all x≠0, is a homomorphism.

Corollary 3. If K is a positive semiring, and A is a regular K-subset of Σ^*, then T (A) is a regular B-subset of Σ^*.

Proof: It follows from the composition of the image automaton and from Proposition 11.

Proposition 12. Let A be a regular B-subsubset of Σ^*. Then, A viewed as an unambiguous K-subsubset of Σ^* is regular.

Proof: Let A be a deterministic Σ-automaton with dynamics A. If A is viewed as a K-Σ-automaton, then its dynamics is viewed as a K-subset of Σ^*.

Remark 6. Corollary 3 and Proposition 12 simultaneously show that for any unambiguous subsets of Σ^*, regularity is independent of the choice of K.

4 The Equality Theorem

In this section it will be assumed that K is a subsemiring of a commutative semiring.

Theorem 3. (The Equality Theorem). Let A_1 and A_2 be recognizable K-subsets of Σ^* and let $A_i=(Q_i,\Sigma_i,E_i,I_i,T_i)$ be K-Σ-automata such that $|A_i|=A_i$, i=1,2. If n_i = Card Q_i and if $A_1(s)=A_2(s)$, $\forall\, s\in\Sigma^*$ such that $|s|<n_1+n_2$, then $A_1=A_2$.

Proof: Consider the automaton $A_1\cup A_2=(Q_1\cup Q_2,\Sigma_1\cup\Sigma_2, E_1\cup E_2,I_1\cup I_2,T_1\cup T_2)$, where Q_1 and Q_2 are assumed disjoint and the transition matrix is From $A_1\cup A_2$ we derive the automata $B_i=(Q_1\cup Q_2,E_1\cup E_2,I_i,T_1\cup T_2)$, i=1,2 and observe that $|B_i|=|A_i|=A_i$, i=1,2. Thus the conclusion of Theorem 3 follows from:

Theorem 4. Let $A_i=(Q_i,\Sigma_i,E_i,I_i,T_i)$, i=1,2 be K-Σ-automata differing only in their initial subsets. Then $|A_1|=|A_2|$ if and only if $|A_1|(s)=|A_2|(s)$, $\forall\, s\in\Sigma^*$ such that $|s|<$ Card Q.

Proof: By assumption K is a subsemiring of a semiring (field F). Thus, we may regard A_i as F-Σ-automata. If $|A_1|$ and $|A_2|$ are equal as F-subsets of Σ^*, then they are also equal as K-subsets. Thus, we may replace K by F, or equivalently assume that K is a field. Thus K^Q is a vector space over K of dimension n=Card Q. Consider the K-Σ-automaton A=(Q,Σ,E,I,T) with $I=I_1\backslash I_2$. Since $|A|(s)\ |A_1|(s)\ \backslash|A_1|(s)$ we have |A|(s)=0 if |s|<n, and we wish to prove that |A|(s)=0, ∀ s ∈ Σ^*. Equivalently, in view of Corollary 1, we are given that (I(s))T=0, if |s|<n and we wish to prove that (I(s))T=0, ∀ s ∈ Σ^*. Define $W=\{X\mid X\in K^Q, XT=0\}$, V_j=subspace of K^Q generated by vectors I(s), with $|s|\le j$. Assuming that n>0, we have $V_0\subset V_1\subset\ldots\subset V_{n-1}\subset W$. The cases I=0 or T=0 may be ruled out since then obviously (I(s))T=0. Therefore, $\dim V_0=1$, dim W=n-1. It follows that for some $0\le j<n-1$, $V_j=V_{j+1}$. Since V_{j+2} is generated by all the vectors X and Xτ, with $X\in V_{j+1}$, it follows that $V_{j+1}=V_{j+2}$ and thus by induction $V_j=V_{j+p}$, $\forall p\ge 0$. This implies $V_j\subset W$, $\forall j\in\mathbb{N}$. Thus, for any ∀ s ∈ Σ^* we have $I(s)\in W$; i.e, (I(s))T=0.

Corolario 4. If A=(Q,Σ,E,I,T) is a K-Σ-automaton, then |A|=0, if and only if, |A|(s)=0, ∀ s ∈ Σ^* such that |s| < Card Q.

The following example shows that the numerical bound in **Theorem 3** is the best possible.

Example 3. Let $0<n_1\le n_2$ be integers. Consider the alphabet Σ consisting of a single letter τ. Let A_1 be the deterministic automaton, with n_1 states represented symbolically

by the path $\rightarrow \mathbf{i} \overset{\tau^{n1-1}}{\rightarrow} \mathbf{t} \rightarrow$ Thus $A_1=\tau^{n1-1}$. Let A_2 be the deterministic complete automaton, with n_2 states represented symbolically by the loop $\rightarrow \mathbf{i} \overset{\tau^{n2}}{\rightarrow} \mathbf{i}$, with $t=i\tau^{n1-1}$ as terminal state. Then $A_2= \tau^{n1-1}(\tau^{n2})$. Thus $\tau^{n1+n2-1} \in A_2 \backslash A_1$. However, for all lower exponents k we have $\tau^k \in A_2$, if and only if, $\tau^k \in A_1$.

5 Conclusion

An important consequence of Theorem 3 is: given any two recognizable K-subsets A_1 and A_2 of Σ^*, it is decidable whether or not $A_1=A_2$. This statement requires several caveats. First the word given should be interpreted to mean that K-Σ-automata $A_i=(Q_i,\Sigma_i,E_i,I_i,T_i)$ such that $|A_i|= A_i$, for i=1,2 are explicitly provided. Then, all the paths of length less than n=Card Q_1+Card Q_2 can be enumerated and thus $|A_1|(s)$ and $|A_2|(s)$ can be computed for $|s| < n$.

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